Deciding All Behavioral Equivalences at Once II
The Silent-Step Spectrum

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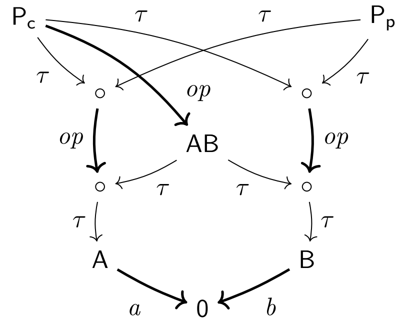
Abstract

We provide the first generalized game characterization of van Glabbeek's linear-time–branching-time spectrum with silent steps. In our quantitative formulation, *one* multi-dimensional energy game can be used to characterize and decide a wide array of weak behavioral equivalences between stability-respecting branching bisimilarity and weak trace equivalence in one go. Deciding all notions this way takes time exponential in the number of states. To establish correctness, we relate attacker-winning energy budgets and distinguishing sublanguages of Hennessy–Milner logic that we characterize by eight dimensions of formula expressiveness.

Keywords: Game theory, Behavioral equivalence, Modal logics

1. Introduction: Mechanizing the Spectrum

Picking the right notion of behavioral equivalence for a particular use case can be hard. For example, Bell [2] has wondered how to precisely justify the equivalence of the states $\mathsf{P_c}$ and $\mathsf{P_p}$ in the following transition system, which originates from a program and its rewriting in the context of a parallelizing compiler:



The solution how to most precisely relate $\mathsf{P_c}$ and $\mathsf{P_p}$ will be given in Example 6.2.

Other researchers have run into similar questions to pick a fitting equivalence for verification and encoding challenges [e.g. 3, 4]. Theoretically, van Glabbeek's "linear-time–branching-time spectrum" [5, 6, 7] brings order to the zoo of equivalences by casting them as a hierarchy of modal logics. But practically, it is difficult to navigate in particular the second part [6], which considers so-called *weak equivalences* that abstract from "internal" behavior, expressed by "silent" τ-steps. Abstracting internal behavior is crucial to model

[⋆] This article is a revised version of "One Energy Game for the Spectrum between Branching Bisimilarity and Weak Trace Semantics" [1] featuring an extended treatment of modal characterizations, full proofs, and a less-exponential variant of the game.

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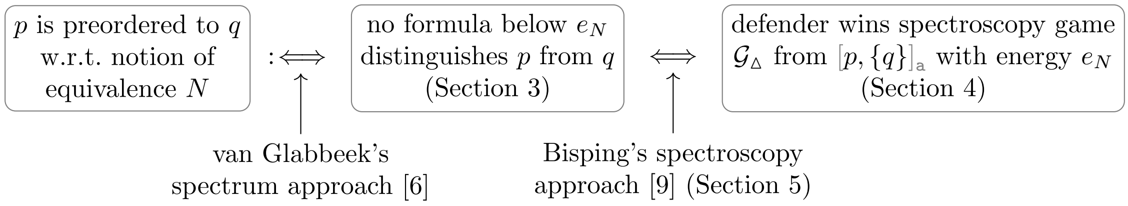


Figure 1: How the paper combines the weak spectrum [6] and the spectroscopy approach [9].

communication happening without participation of the observer and to inspect refinements, that is, for virtually every application.

In this paper, we *operationalize the silent-step spectrum* of van Glabbeek's "linear-time–branching-time spectrum II" [6]. The resulting algorithm enables researchers, when they provide a set of processes that ought to be equated (or distinguished) for their scenario, to learn which (in-)equivalences in the spectrum match this set. Implicitly, we obtain decision procedures (and games) for each individual notion of equivalence as a by-product. This continues our project of *deciding all behavioral equivalences at once,* which began with *a game for linear-time–branching-time spectroscopy* [8]. There, we solved the problem for van Glabbeek's strong spectrum without silent steps [5]. So, the present article can be understood as a second part to [8], analogously to van Glabbeek's two publications on the spectrum.

As outlined in Figure 1, we apply our improved recent approach [9] to use a *generalized bisimulation energy game* with moves corresponding to sets of conceivable distinguishing formulas. The game is a *multi-weighted energy game* [10, 11, 12] where moves use up attacker's resources to distinguish processes, which correspond to syntactic features of Hennessy–Milner logic (HML) formulas. Thereby, defender-won energy levels reveal non-distinguishing subsets of HML and thus sets of maintained equivalences.

Applying the above approach to the weak spectrum faces many obstacles: The modal logics of the weak spectrum in [6] are quite intricate and are not closed under HML-subterms. Also, van Glabbeek [6] does not account for unstable linear-time equivalences, but other publications like Gazda et al. [13] use these. On the game side, existing weak bisimulation games by De Frutos Escrig et al. [14] and Bisping et al. [15] lack moves for many observations that are relevant for weaker notions in the spectrum. This paper shows how all this can still be brought together.

Contributions. At its core, this article leverages the spectroscopy energy game approach of [9] with modalities needed to cover the weak equivalence spectrum of [6], namely, delayed observations, stable conjunctions, and branching conjunctions. More precisely:

- In Section 3, we capture a big chunk of the *linear-time–branching-time spectrum with silent steps by measuring expressive powers* used in HML-subsets, which we prove to correspond to a hierarchy of notions between stability-respecting branching bisimilarity and weak trace equivalence.
- In Section 4, we introduce the first generalized game characterization of the silent-step equivalence spectrum. For this, we adapt the *spectroscopy energy game* of [9] to account for distinctions in terms of delayed observations ($\langle\varepsilon\rangle\langle a\rangle\ldots$), stable conjunctions ($\langle\varepsilon\rangle\bigwedge\{\neg\langle\tau\rangle\mathsf{T},\ldots\}$), and branching conjunctions ($\langle\varepsilon\rangle\bigwedge\{(a)\ldots,\langle\varepsilon\rangle\ldots\}$).
- Section 5 proves that *winning energy levels and equivalences coincide* by closely relating distinguishing formulas and ways the attacker may win the energy game.
- Section 6 lays out how to utilize the game for everyday research to *decide all equivalences at once* using our prototype tool and discusses application to further equivalences.
- Section 7 treats the *exponential complexity of the algorithm,* and shows how a less-exponential game can be employed to alleviate the worst part of it.

The work is framed by some preliminaries in Section 2 and a discussion of related work in Section 8.

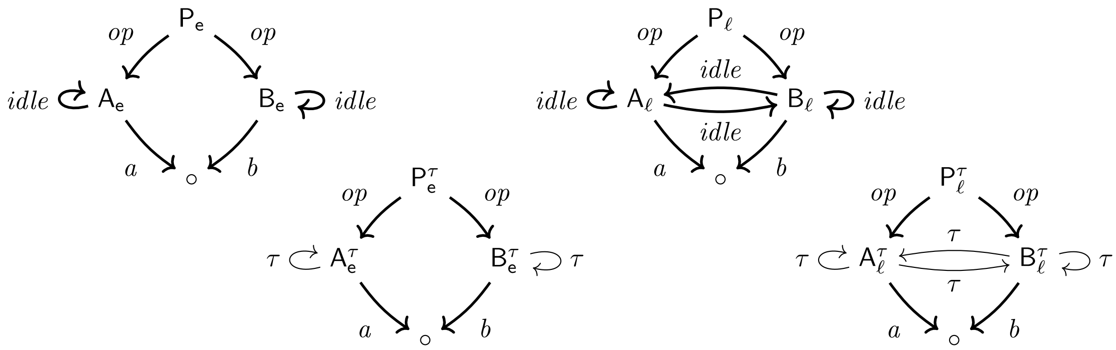


Figure 2: A pair of processes $\mathsf{P_e}$ and P_ℓ, together with variants $\mathsf{P}^\tau_\mathsf{e}$ and P^τ_ℓ where *idle* has been abstracted into internal τ-behavior.

Formalization. The core results of this article have been formalized in Isabelle/HOL. The theory can be found on the Archive of Formal Proofs [16]. Important lemmas come with a 🔗-footnote pointing to the corresponding Isabelle facts (mirrored on `https://proofs.equiv.io/AFP/Weak_Spectroscopy/`).

What's New? Compared to the EXPRESS/SOS'24 version of this work [1], the biggest new contribution is the deep treatment of modal characterizations (Section 3). Also, Section 6 has been extended, and most of Section 7 is new material, providing improved complexity bounds using a better game algorithm and a trick to reduce the game's branching degree. Moreover, we include all interesting proofs and links to the Isabelle/HOL formalization, as well as some small corrections.

Context. This article is part of a series of works on deciding all behavioral equivalences at once. At points, this makes it quite technical. For a more didactic presentation of the underlying ideas, we refer to the first author's PhD thesis "Generalized Equivalence Checking of Concurrent Programs" [17].

2. Distinctions and Equivalences in Systems with Silent Steps

This paper follows the paradigm that *equivalence is the absence of possibilities to distinguish.* Equivalently, one could speak about apartness, i.e. the view that non-equivalence is based on evidence of difference [18]. Technically, this view allows to characterize most equivalences through inductive definitions, without resorting to coinduction. We begin by introducing the things we want to equate or distinguish: states in transition systems (Subsection 2.1). Then, we define how to express distinctions: by Hennessy–Milner logic formulas (Subsection 2.2).

2.1. Transition Systems and Equivalences

Definition 2.1 (Labeled transition system with silent steps)**.** A *labeled transition system* is a tuple $\mathcal{S} = (\mathcal{P}, \Sigma, \rightarrow)$ where $\mathcal{P}$ is the set of *processes,* Σ is the set of *actions,* and $\rightarrow\ \subseteq \mathcal{P} \times \Sigma \times \mathcal{P}$ is the *transition relation.*

$\tau \in \Sigma$ labels *silent steps* and $\twoheadrightarrow$ is notation for the reflexive transitive closure of *internal activity* $\xrightarrow{\tau}{}^*$. The name $\varepsilon \notin \Sigma$ is reserved and indicates no (visible) action. A process p is called *stable* if $p \overset{\tau}{\not\rightarrow}$. We write $p \xrightarrow{(\alpha)} p'$ if $p \xrightarrow{\alpha} p'$, or if $\alpha = \tau$ and $p = p'$.

We implicitly lift the relations to sets of processes $P \xrightarrow{\alpha} P'$ (with $P, P' \subseteq \mathcal{P}$, $\alpha \in \Sigma$), which is defined to be true if $P' = \{p' \in \mathcal{P} \mid \exists p \in P.\, p \xrightarrow{\alpha} p'\}$.

Example 2.1. Figure 2 presents transition systems of four processes: $\mathsf{P_e}$ makes a nondeterministic choice *op* between a and b, performing arbitrarily many *idle*-actions in between. P_ℓ does the same but can change the choice while idling. $\mathsf{P}^\tau_\mathsf{e}$ and P^τ_ℓ are variants of the two obtained by abstracting *idle* into τ-actions.

The example is helpful to test whether a process equivalence can be a congruence for abstraction. Any congruence for abstraction $\sim$ would need to have the property that $\mathsf{P_e} \sim \mathsf{P}_\ell$ implies $\mathsf{P_e^\tau} \sim \mathsf{P}_\ell^\tau$. So, if we just had a quick way of *testing for all weak behavioral equivalences at once,* we could quickly narrow down which equivalences work for this example. Incidentally, this paper's spectroscopy algorithm can do precisely this: decide all equivalences at once.

In the next section, we will encounter many behavioral equivalences for systems with internal behavior. As a first taste, let us look at the coarsest and the finest ones, and how they are commonly defined in the literature.

The coarsest common notion of equivalence for systems with silent steps is *weak trace equivalence.* It can be understood as a form of language equivalence on the words that can be observed starting from states, where τ-steps are skipped.

Definition 2.2 (Word steps)**.** We extend the notion $\twoheadrightarrow$ from τ-sequences to words $\vec{w} \in \Sigma^*$ as follows:

- If $\vec{w}$ is the empty word λ, we have $p \overset{\vec{w}}{\twoheadrightarrow} p'$ iff $p \twoheadrightarrow p'$.
- If $\vec{w} = \vec{w}'\tau$, we have $p \overset{\vec{w}}{\twoheadrightarrow} p'$ iff $p \overset{\vec{w}'}{\twoheadrightarrow} p'$.
- If $\vec{w} = \vec{w}'a$, we have $p \overset{\vec{w}}{\twoheadrightarrow} p''$ iff there exists p' such that $p \overset{\vec{w}'}{\twoheadrightarrow} p' \xrightarrow{a}\twoheadrightarrow p''$.

Definition 2.3 (Weak trace preorder and equivalence)**.** Let $\mathsf{wtraces}(p)$ be the set of $\vec{w} \in \Sigma^*$ such that there is some p' with $p \overset{\vec{w}}{\twoheadrightarrow} p'$.

For two processes p and q, we say that p is *weakly trace-preordered to* q iff $\mathsf{wtraces}(p) \subseteq \mathsf{wtraces}(q)$.

Moreover, precisely if $\mathsf{wtraces}(p) = \mathsf{wtraces}(q)$, then p and q are *weakly trace-equivalent.*

In Example 2.1, $\mathsf{P_e}$ and P_ℓ are weakly trace-equivalent because $\mathsf{wtraces}(\mathsf{P_e}) = \mathsf{wtraces}(\mathsf{P}_\ell)$. $\mathsf{P_e^\tau}$ and P_ℓ^τ are weakly trace-equivalent as well. Also, $\mathsf{P_e^\tau}$ is weakly trace-preordered to $\mathsf{P_e}$, but not the other way around because of $(op, idle) \in \mathsf{wtraces}(\mathsf{P_e}) \setminus \mathsf{wtraces}(\mathsf{P_e^\tau})$.

The finest common notion of equivalence for systems with silent steps is stability-respecting branching bisimilarity. Let us quickly recall its operational definition (for instance from [19]); for a visualization, see Figure 3:

Definition 2.4 (Branching bisimilarity, operationally; stability-respecting relations)**.** A symmetric relation $\mathcal{R}$ is a *branching bisimulation* if, for all $(p, q) \in \mathcal{R}$, a step $p \xrightarrow{\alpha} p'$ implies (1) $\alpha = \tau$ and $(p', q) \in \mathcal{R}$, or (2) $q \twoheadrightarrow q' \xrightarrow{\alpha} q''$ for some q', q'' with $(p, q') \in \mathcal{R}$ and $(p', q'') \in \mathcal{R}$.

If moreover for every $(p, q) \in \mathcal{R}$ with $p \overset{\tau}{\not\rightarrow}$, there is some q' with $q \twoheadrightarrow q' \overset{\tau}{\not\rightarrow}$ and $(p, q') \in \mathcal{R}$, the relation is *stability-respecting.*

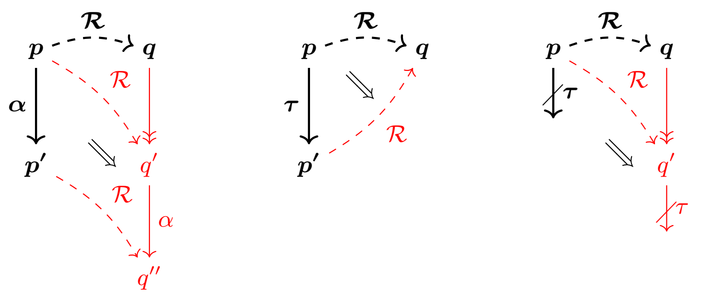
If there is a stability-respecting branching bisimulation $\mathcal{R}_{BB^{sr}}$ with $(p_0, q_0) \in \mathcal{R}_{BB^{sr}}$, then p_0 and q_0 are stability-respecting branching bisimilar.

For instance, A_ℓ^τ and B_ℓ^τ of Example 2.1 are (stability-respecting) branching bisimilar as $\mathcal{R} := \{(\mathsf{A}_\ell^\tau, \mathsf{B}_\ell^\tau), (\mathsf{B}_\ell^\tau, \mathsf{A}_\ell^\tau), (\mathsf{A}_\ell^\tau, \mathsf{A}_\ell^\tau), (\mathsf{B}_\ell^\tau, \mathsf{B}_\ell^\tau), (\circ, \circ)\}$ is a stability-respecting branching bisimulation.

Neither $\mathsf{P_e}$ and P_ℓ, nor $\mathsf{P_e^\tau}$ and P_ℓ^τ are branching bisimilar. The reason is that they each can do transitions to states that allow different weak traces than the other process. But the inequivalence of states can more easily be discussed through the lens of modal characterizations.

2.2. Hennessy–Milner Logic for Branching Bisimilarity

Bisimilarity and other notions of equivalence can conveniently be defined in terms of Hennessy–Milner logic. We direct our attention to variants that allow for silent behavior to happen before visible actions are observed. We thus focus on the following variant, where the brick-red part represents *stable conjunctions* and the steel-blue part *branching conjunctions:*



Standard case of branching simulation | τ-alternative of branching simulation | Respecting stability

Figure 3: Branching simulation and respect for stability in Definition 2.4 (the thick black part implies the thin red part).

Definition 2.5 (Branching Hennessy–Milner logic)**.** We define *stability-respecting branching* Hennessy–Milner modal logic, $\mathsf{HML}_{\text{srbb}}$, over an alphabet of actions Σ by the following context-free grammar starting with φ:

$$
\begin{array}{llll}
\mathsf{HML}_{\text{srbb}}: & \varphi ::= \langle\varepsilon\rangle\chi & & \text{“delayed observation”} \\
& \quad\mid \ \bigwedge\{\psi,\psi,...\} & & \text{“immediate conjunction”} \\
& \chi ::= \langle a\rangle\varphi & \text{with } a \in \Sigma \setminus \{\tau\} & \text{“observation”} \\
& \quad\mid \ \bigwedge\{\psi,\psi,...\} & & \text{“standard conjunction”} \\
& \quad\mid \ \bigwedge\{\neg\langle\tau\rangle\mathsf{T},\psi,\psi,...\} & & \text{“stable conjunction”} \\
& \quad\mid \ \bigwedge\{(\alpha)\varphi,\psi,\psi,...\} & \text{with } \alpha \in \Sigma & \text{“branching conjunction”} \\
& \psi ::= \neg\langle\varepsilon\rangle\chi \mid \langle\varepsilon\rangle\chi & & \text{“negative / positive conjuncts”}
\end{array}
$$

$\bigwedge\{\psi,\psi,...\}$ in the grammar stands for conjunction with arbitrary branching, which includes infinite conjunctions. We write T for the empty conjunction $\bigwedge\varnothing$.

Its semantics $[\![\cdot]\!]^{\mathcal{S}} \colon \mathsf{HML}_{\text{srbb}} \to \mathbf{2}^{\mathcal{P}}$, where a formula “is true,” over a transition system $\mathcal{S} = (\mathcal{P}, \Sigma, \rightarrow)$ is defined in mutual recursion with helper functions $[\![\cdot]\!]_{\varepsilon}$ for subformulas in the “delayed” context (χ-productions) and $[\![\cdot]\!]_{\wedge}$ for conjuncts (ψ-productions):

$$
\begin{aligned}
[\![\langle\varepsilon\rangle\chi]\!]^{\mathcal{S}} &\coloneqq \{p \in \mathcal{P} \mid \exists p' \in [\![\chi]\!]^{\mathcal{S}}_{\varepsilon}.\, p \twoheadrightarrow p'\} \\
[\![\textstyle\bigwedge\Psi]\!]^{\mathcal{S}} \coloneqq [\![\textstyle\bigwedge\Psi]\!]^{\mathcal{S}}_{\varepsilon} &\coloneqq \textstyle\bigcap\{[\![\psi]\!]^{\mathcal{S}}_{\wedge} \mid \psi \in \Psi\} \\
[\![\langle a\rangle\varphi]\!]^{\mathcal{S}}_{\varepsilon} &\coloneqq \{p \in \mathcal{P} \mid \exists p' \in [\![\varphi]\!]^{\mathcal{S}}.\, p \xrightarrow{a} p'\} \\
[\![\neg\langle\tau\rangle\mathsf{T}]\!]^{\mathcal{S}}_{\wedge} &\coloneqq \{p \in \mathcal{P} \mid p \stackrel{\tau}{\not\rightarrow}\} \\
[\![(\alpha)\varphi]\!]^{\mathcal{S}}_{\wedge} &\coloneqq \{p \in \mathcal{P} \mid \exists p' \in [\![\varphi]\!]^{\mathcal{S}}.\, p \xrightarrow{(\alpha)} p'\} \\
[\![\neg\langle\varepsilon\rangle\chi]\!]^{\mathcal{S}}_{\wedge} &\coloneqq \mathcal{P} \setminus [\![\langle\varepsilon\rangle\chi]\!]^{\mathcal{S}}_{\wedge} \\
[\![\langle\varepsilon\rangle\chi]\!]^{\mathcal{S}}_{\wedge} &\coloneqq [\![\langle\varepsilon\rangle\chi]\!]^{\mathcal{S}}
\end{aligned}
$$

Definition 2.6 (Distinguishing formulas)**.** A formula $\varphi \in \mathsf{HML}_{\text{srbb}}$ is said to *distinguish* a process p from q iff $p \in [\![\varphi]\!]^{\mathcal{S}}$ and $q \notin [\![\varphi]\!]^{\mathcal{S}}$. The formula is said to *distinguish* a process p from a set of processes Q iff it is true for p and false for every $q \in Q$.

Example 2.2. In Example 2.1, $\varphi_\tau := \langle\varepsilon\rangle\langle op\rangle\langle\varepsilon\rangle\bigwedge\{\neg\langle\varepsilon\rangle\langle b\rangle\mathsf{T}\}$ distinguishes $\mathsf{P}^\tau_\mathsf{e}$ from P^τ_ℓ. Formula φ_τ states that a weak *op*-step may happen such that, afterwards, b is not τ-reachable. This is true of $\mathsf{P}^\tau_\mathsf{e}$ because of the $\mathsf{A}^\tau_\mathsf{e}$-state, but not of P^τ_ℓ.

The name already alludes to $\mathsf{HML}_{\text{srbb}}$ as a whole characterizing stability-respecting branching bisimilarity. The power of Definition 2.5 to distinguish mirrors exactly the power of Definition 2.4 to equate:

Lemma 2.1. $\mathsf{HML}_{\text{srbb}}$ *characterizes stability-respecting branching bisimilarity.*[1]

This kind of characterization result between a modal logic and a notion of equivalence is usually referred to as a *Hennessy–Milner theorem*. We will prove it as Lemma 3.3, together with a few similar facts.

Coarser notions of equivalence can then be characterized through subsets of the logic. For instance, we can mirror Definition 2.3 as follows:

Lemma 2.2. *Consider the subset* $\mathcal{O}_\mathrm{T} \subseteq \mathsf{HML}_{\text{srbb}}$ *given by the following grammar:*

$$\mathcal{O}_\mathrm{T} : \qquad \varphi_\mathrm{T} \quad ::= \quad \langle\varepsilon\rangle\langle a\rangle\varphi_\mathrm{T} \quad | \quad \langle\varepsilon\rangle\mathsf{T} \quad | \quad \mathsf{T} \qquad \text{with } a \in \Sigma \setminus \{\tau\}.$$

There is a formula $\varphi \in \mathcal{O}_\mathrm{T}$ *distinguishing* p *from* q *precisely if* p *is* not *trace-preordered to* q.[2]

Proof. Clearly, for a word $\vec{w} = a_1 \dots a_n \in (\Sigma \setminus \{\tau\})^*$, the formula $\langle\varepsilon\rangle\langle a_1\rangle \dots \langle\varepsilon\rangle\langle a_n\rangle\langle\varepsilon\rangle\mathsf{T} \in \mathcal{O}_\mathrm{T}$ is true for a state p precisely if there is a p' such that $p \overset{\vec{w}}{\twoheadrightarrow} p'$, that is, if $\vec{w} \in \mathsf{wtraces}(p)$. As $\mathsf{wtraces}(p) = \mathsf{wtraces}(q)$ for states p and q precisely if $\mathsf{wtraces}(p) \cap (\Sigma \setminus \{\tau\})^* = \mathsf{wtraces}(q) \cap (\Sigma \setminus \{\tau\})^*$, this completes the proof. □

Remark 2.1. Definition 2.5 is constructed to fit the distinctive powers we need from HML to characterize varying notions of the weak spectrum by controlling which productions are used. Subformulas in the grammar usually start with $\langle\varepsilon\rangle \dots$, effectively hiding silent steps. Productions that combine operators without $\langle\varepsilon\rangle$-positions increase additional distinctive power, as they measure the absence of ε-steps. We will use immediate conjunctions to distinguish non-delay-bisimilar processes, and branching conjunctions (that contain one positive conjunct without leading $\langle\varepsilon\rangle$) to distinguish non-$\eta$-(bi)similar processes. Allowing the observation of stabilization, $\neg\langle\tau\rangle\mathsf{T}$, increases distinctive power; *requiring* stabilization for conjunct observations (i.e. disallowing other conjunctions) decreases it.

3. Recharting the Weak Spectrum of Behavioral Equivalences

As our first main contribution, we provide a variation of van Glabbeek's *linear-time–branching-time spectrum* part II for systems with silent steps [6]. While his part I on notions ignoring silent steps has seen a journal version [7] and refinements by others [20], part II has only been published as "extended abstract" [6] (accompanied by a "preliminary version" on van Glabbeek's website reporting some proofs, but cursory in parts).

So, in this section, we introduce our version of the silent-step spectrum in Subsection 3.1, and then do the legwork of relating it to common definitions of the notions with proofs for all interesting cases.

Our main goal is to capture the spectrum quantitatively such that it works well with our game approach of the next section. But our variant of the weak spectrum is also interesting in its own right as we use $\mathsf{HML}_{\text{srbb}}$, which is a more-standard Hennessy–Milner logic than the one of [6]. Moreover, we increase clarity by leaving aside many variants of notions from [6], and, on the other hand, cater for some additional ones such as (im-)possible futures and (unstable) failures.

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[1] lemma `Branching_Bisimilarity.lts_tau.sr_branching_bisim_is_hmlsrbb`
[2] lemma `Weak_Traces.lts_tau.trace_equals_trace_to_formula`

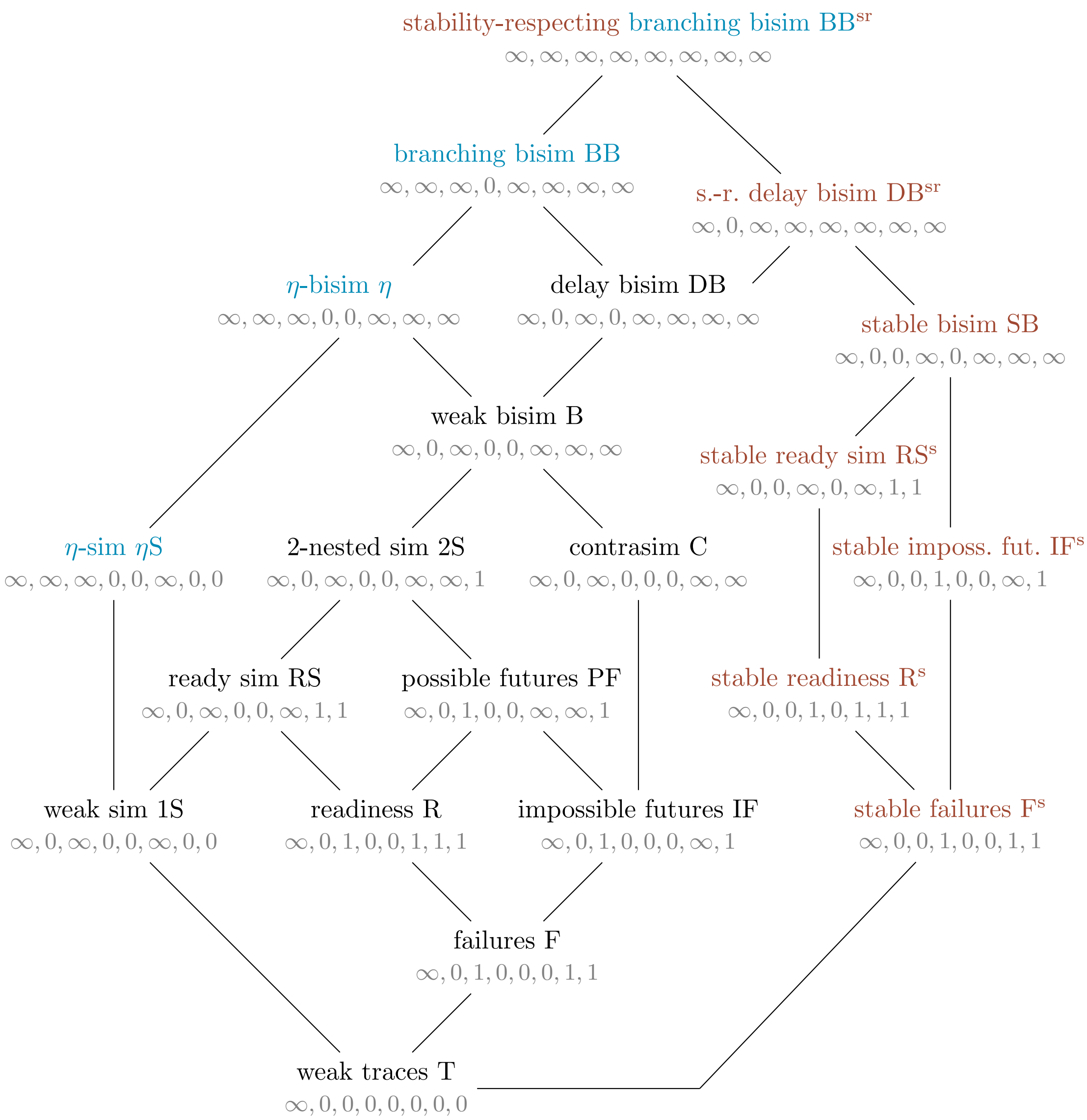

Figure 4: Hierarchy of weak behavioral equivalences/preorders, becoming finer towards the top. Each notion N comes with its expressiveness coordinate e_N.

3.1. The Silent-Step Spectrum

At the core, we follow van Glabbeek's [6] approach to define partially ordered "notions of observability," which entail behavioral preorders and equivalences given as modal characterizations.

Definition 3.1 (Equivalence spectra and preorders)**.** An *equivalence spectrum* $(\mathbf{N}, \leq, \mathcal{O}_{N \in \mathbf{N}})$ consists of

- notions of observability, $\mathbf{N}$, partially ordered by $\leq \subseteq \mathbf{N} \times \mathbf{N}$, and
- corresponding logics $\mathcal{O}_N : \mathbf{2}^{\mathsf{HML}_{\text{srbb}}}$ for $N \in \mathbf{N}$.

$\mathcal{O}_{N \in \mathbf{N}}$ must be monotonic, that is: for any two notions $N, M \in \mathbf{N}$, it holds that $N \leq M$ implies $\mathcal{O}_N \subseteq \mathcal{O}_M$.

A sublogic, $\mathcal{O}_N \subseteq \mathsf{HML}_{\text{srbb}}$, corresponding to a notion of observability N, *distinguishes* two processes, $p \not\preceq_N q$, if there is $\varphi \in \mathcal{O}_N$ with $p \in [\![\varphi]\!]^{\mathcal{S}}$ and $q \notin [\![\varphi]\!]^{\mathcal{S}}$. Otherwise N *preorders* them, $p \preceq_N q$. If processes are mutually N-preordered, $p \preceq_N q$ and $q \preceq_N p$, then they are considered *N-equivalent*, $p \sim_N q$.

The monotonicity of the definition yields that "upper" notions entail "lower" ones:

Lemma 3.1. *If $N \leq M$, then $p \preceq_M q$ implies $p \preceq_N q$.*

Van Glabbeek [6] uses about 20 binary dimensions to characterize 155 "notions of observability" (derived from five dimensions of testing scenarios).

In our version, we capture the *notions of observability as coordinates* in a quantitative 8-dimensional space of syntactic features in HML formulas.

We will "price the expressiveness" of $\mathsf{HML}_{\text{srbb}}$-formulas by vectors we call *energies*. The pricing allows to conveniently select subsets $\mathcal{O}_N \subseteq \mathsf{HML}_{\text{srbb}}$ in terms of coordinates.

Definition 3.2 (Energies)**.** We denote as *energies*, $\mathbf{En}_\infty$, the set $(\mathbb{N} \cup \{\infty\})^8$.

We compare energies component-wise: $(e_1, \dots, e_8) \leq (f_1, \dots, f_8)$ iff $e_i \leq f_i$ for each i. Least upper bounds $\sup \{\dots\}$ are defined as usual as component-wise supremum.

We write $\hat{\mathbf{e}}_i$ for the standard unit vector where the i-th component is 1 and every other component equals 0. The vector $(0, 0, \dots, 0)$ is written as $\mathbf{0}$. Vector addition and subtraction happen component-wise, as usual.

In Figure 4, we order weak equivalences along dimensions of $\mathsf{HML}_{\text{srbb}}$-expressiveness in terms of *operator depths* (i.e. maximal occurrences of an operator on a path from root to leaf in the abstract syntax tree). Intuitively, the dimensions are:

1. Modal depth (of observations $\langle a \rangle$, (α)),
2. Depth of branching conjunctions (with one (α)-observation conjunct that does not start with $\langle \varepsilon \rangle$),
3. Depth of unstable conjunctions (that do not enforce stability by a $\neg\langle\tau\rangle\mathsf{T}$-conjunct),
4. Depth of stable conjunctions (that do enforce stability by a $\neg\langle\tau\rangle\mathsf{T}$-conjunct),
5. Depth of immediate conjunctions (that are not preceded by $\langle \varepsilon \rangle$),
6. Maximal modal depth of positive conjuncts in conjunctions,
7. Maximal modal depth of negative conjuncts in conjunctions,
8. Depth of negations.

Definition 3.3 (Formula prices)**.** The *expressiveness price* of a formula $\mathsf{expr}\colon \mathsf{HML}_{\text{srbb}} \to \mathbf{En}_\infty$ is defined in mutual recursion with helper functions $\mathsf{expr}^\varepsilon$ and $\mathsf{expr}^\wedge$; if multiple rules apply to a formula (e.g. because T equals $\bigwedge\varnothing$), pick the first one:

$$
\begin{aligned}
\mathsf{expr}(\mathsf{T}) := \mathsf{expr}^\varepsilon(\mathsf{T}) &:= \mathbf{0}\\
\mathsf{expr}(\langle\varepsilon\rangle\chi) &:= \mathsf{expr}^\varepsilon(\chi)\\
\mathsf{expr}(\textstyle\bigwedge\Psi) &:= \hat{\mathbf{e}}_5 + \mathsf{expr}^\varepsilon(\textstyle\bigwedge\Psi)\\
\mathsf{expr}^\varepsilon(\langle a\rangle\varphi) &:= \hat{\mathbf{e}}_1 + \mathsf{expr}(\varphi)\\
\mathsf{expr}^\varepsilon(\textstyle\bigwedge\Psi) &:= \sup\{\mathsf{expr}^\wedge(\psi) \mid \psi \in \Psi\} + \begin{cases} \hat{\mathbf{e}}_4 & \text{if } \neg\langle\tau\rangle\mathsf{T} \in \Psi\\ \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 & \text{if there is } (\alpha)\varphi \in \Psi\\ \hat{\mathbf{e}}_3 & \text{otherwise}\end{cases}\\
\mathsf{expr}^\wedge(\neg\langle\tau\rangle\mathsf{T}) &:= (0,0,0,0,0,0,0,1)\\
\mathsf{expr}^\wedge(\neg\varphi) &:= \sup\{\hat{\mathbf{e}}_8 + \mathsf{expr}(\varphi),\quad (0,0,0,0,0,0,(\mathsf{expr}(\varphi))_1,0)\}\\
\mathsf{expr}^\wedge((\alpha)\varphi) &:= \sup\{\hat{\mathbf{e}}_1 + \mathsf{expr}(\varphi),\quad (0,0,0,0,0,1+(\mathsf{expr}(\varphi))_1,0,0)\}\\
\mathsf{expr}^\wedge(\varphi) &:= \sup\{\qquad \mathsf{expr}(\varphi),\quad (0,0,0,0,0,(\mathsf{expr}(\varphi))_1,0,0)\}
\end{aligned}
$$

Especially the $\mathsf{expr}^\wedge$-cases might require some explanation. They take the supremum between the intrinsic expressiveness price of the conjunct and an expression that copies the modal depth of the conjunct to dimension 6 for positive conjuncts and dimension 7 for negative ones. These dimensions thus keep track of the bounds of how far into possible and impossible futures conjunctions may peek.

Definition 3.4 (Linear-time–branching-time equivalences)**.** Each notion N named in Figure 4 with coordinate e_N is defined through the language of formulas with prices below, i.e., through $\mathcal{O}_N = \{\varphi \mid \mathsf{expr}(\varphi) \leq e_N\}$.

Because of Definition 3.1, this means that we set $p \preceq_N q$ with respect to notion N iff no φ with $\mathsf{expr}(\varphi) \leq e_N$ distinguishes p from q. Basically, the coordinates restrict the use of certain operators in formulas that are allowed to distinguish processes. Low components "crop" parts of the grammar of Definition 2.5. So, this paper sees notions of preorder and equivalence to be primarily defined modally through these coordinates—and not through other characterizations such as trace sets or state relations.

Example 3.1. The formula $\varphi_\tau = \langle\varepsilon\rangle\langle op\rangle\langle\varepsilon\rangle\bigwedge\{\neg\langle\varepsilon\rangle\langle b\rangle\mathsf{T}\}$ in Example 2.2 has expressiveness price $\mathsf{expr}(\varphi_\tau) = (2, 0, 1, 0, 0, 0, 1, 1)$. The coordinate is below the one of failures $e_\mathrm{F} = (\infty, 0, 1, 0, 0, 0, 1, 1)$ in Figure 4. Accordingly, $\mathsf{P}^\tau_\mathsf{e}$ is distinguished from P^τ_ℓ by the failure observation $\varphi_\tau \in \mathcal{O}_\mathrm{F}$, that is, $\mathsf{P}^\tau_\mathsf{e} \not\preceq_\mathrm{F} \mathsf{P}^\tau_\ell$. There are neither strictly-stable nor strictly-positive formulas to distinguish $\mathsf{P}^\tau_\mathsf{e}$ from P^τ_ℓ. Therefore, stable bisimulation preorder, $\mathsf{P}^\tau_\mathsf{e} \preceq_\mathrm{SB} \mathsf{P}^\tau_\ell$, and η-simulation preorder, $\mathsf{P}^\tau_\mathsf{e} \preceq_{\eta\mathrm{S}} \mathsf{P}^\tau_\ell$, apply. (The latter implies the more well-known weak simulation preorder.)

For stability-respecting branching bisimilarity, nothing is cropped. Thus, $\mathcal{O}_{\mathrm{BB}^\mathrm{sr}} = \mathsf{HML}_\mathrm{srbb}$ means that Lemma 2.1 establishes that our modal characterization corresponds to the common operational definition. For some notions, there are superficial differences to other modal characterizations in the literature. We devote the remainder of the section to prove that we still characterize the same behavioral equivalences. But first, we give two examples of the nature of the differences.

Example 3.2 (Weak trace equivalence and preorder)**.** The notion of weak trace preorder (and equivalence) is defined through $e_\mathrm{T} = (\infty, 0, 0, 0, 0, 0, 0, 0)$ and Definition 3.3 inducing exactly the language $\mathcal{O}_\mathrm{T}$ of Lemma 2.2.

Note that $\mathcal{O}_\mathrm{T}$ slightly deviates from languages one would find in other publications. For instance, Gazda et al. [13] do not have the second production. But this production does not increase expressiveness, as $[\![\langle\varepsilon\rangle\mathsf{T}]\!] = [\![\mathsf{T}]\!] = \mathcal{P}$.

Example 3.3 (Weak bisimulation equivalence and preorder)**.** The logic of weak bisimulation observations $\mathcal{O}_\mathrm{B}$ defined through $e_\mathrm{B} = (\infty, 0, \infty, 0, 0, \infty, \infty, \infty)$ equals the language defined by the grammar:

$$\begin{array}{llll} \mathcal{O}_\mathrm{B}: & \varphi_\mathrm{B} & ::= & \langle\varepsilon\rangle\chi_\mathrm{B} \mid \mathsf{T} \\ & \chi_\mathrm{B} & ::= & \langle a\rangle\varphi_\mathrm{B} \mid \bigwedge\{\psi_\mathrm{B}, \psi_\mathrm{B}, \ldots\} \\ & \psi_\mathrm{B} & ::= & \neg\langle\varepsilon\rangle\chi_\mathrm{B} \mid \langle\varepsilon\rangle\chi_\mathrm{B}. \end{array}$$

Let us contrast this to the definition for weak bisimulation observations $\mathcal{O}_{\mathrm{B}'}$ from Gazda et al. [13]:

$$\mathcal{O}_{\mathrm{B}'}: \quad \varphi_{\mathrm{B}'} \quad ::= \quad \langle\varepsilon\rangle\varphi_{\mathrm{B}'} \mid \langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\varphi_{\mathrm{B}'} \mid \bigwedge\{\varphi_{\mathrm{B}'}, \varphi_{\mathrm{B}'}, \ldots\} \mid \neg\varphi_{\mathrm{B}'}.$$

Both languages put $\langle\varepsilon\rangle$-delays *before* observations. The conceptual difference is that $\mathcal{O}_\mathrm{B}$ requires $\langle\varepsilon\rangle$-delays *before conjunctions* (and negations), while $\mathcal{O}_{\mathrm{B}'}$ requires them *after observations.* In this sense, $\mathcal{O}_\mathrm{B}$ is more *uniform* in how it weakens strong bisimulation observations than its siblings in the literature.

Our $\mathcal{O}_\mathrm{B}$ allows a few formulas that $\mathcal{O}_{\mathrm{B}'}$ lacks, e.g. $\langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\mathsf{T}$. This does not add expressiveness as $\mathcal{O}_{\mathrm{B}'}$ has $\langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\mathsf{T}$ and $[\![\langle\varepsilon\rangle\langle\varepsilon\rangle\varphi]\!] = [\![\langle\varepsilon\rangle\varphi]\!]$.

For the other direction, there is a bigger difference due to $\mathcal{O}_{\mathrm{B}'}$ allowing more freedom in the placement of conjunction and negation. In particular, it permits top-level conjunctions and negated conjunctions without $\langle\varepsilon\rangle$ in between. But these features do not add distinctive power. $\mathcal{O}_{\mathrm{B}'}$ additionally allows top-level negation, and this adds distinctive power to the preorders, effectively turning them into equivalence relations. We do not enforce this and thus our $\preceq_\mathrm{B} \neq \sim_\mathrm{B}$; for instance, $\tau.a \preceq_\mathrm{B} \tau + \tau.a$, but $\tau + \tau.a \not\preceq_\mathrm{B} \tau.a$ due to $\langle\varepsilon\rangle\bigwedge\{\neg\langle\varepsilon\rangle\langle a\rangle\mathsf{T}\}$. (It behaves like weak simulation, but only with respect to *initial* internal behavior!) However, if a formula $\neg\varphi \in \mathcal{O}_{\mathrm{B}'}$ distinguishes p from q, then φ distinguishes q from p, and a formula equivalent to one of these two is in $\mathcal{O}_\mathrm{B}$. So this difference is ironed out once we consider the equivalence $\sim_\mathrm{B} = \preceq_\mathrm{B} \cap \succeq_\mathrm{B}$. In Lemma 3.3, we will prove $\sim_\mathrm{B}$ to coincide with the usual weak bisimilarity.

Remark 3.1. None of the logics in Figure 4 restrict the first dimension, but the modal depth is kept to simplify the calculation of dimensions 6 and 7. It could also be used to define k-step bisimilarity and similar notions. For a detailed discussion of common questions about such spectra, see [17, Subsection 3.2.3f.].

In general, there is no deeper necessity to use *exactly* the dimensions that this paper employs or the original ones of [6]—in both cases, they are chosen in order to conveniently cover notions of equivalence that stem from varying contexts. To address even more notions, additional dimensions would be necessary, as we will discuss in Subsection 6.3.

Remark 3.2. Compared with the workshop version of this paper [1], we have slightly modified Definition 3.3 to have $\mathsf{expr}(\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\}) = (0,0,0,1,0,0,0,1)$. Previously, we had priced such "empty" stable conjunctions at $(0,0,0,1,0,0,0,0)$. The previous definition included observations of "failures to stabilize" like $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}, \neg\langle a\rangle\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\}\}$ in the language of stable ready simulation, which is incorrect. The change prevents this kind of problem.

3.2. Abstractions of Simulation

In Definition 3.4, there are two abstractions of the common strong simulation equivalence for systems with silent steps. We call a weak notion an abstraction of a strong one if the two coincide on transition systems without τ-transitions.

Definition 3.5 (Weak simulation)**.** A relation $\mathcal{R}$ is a *weak simulation* if, for all $(p,q) \in \mathcal{R}$, a step $p \xrightarrow{\alpha} p'$ implies (1) $\alpha = \tau$ and $(p',q) \in \mathcal{R}$, or (2) $q \twoheadrightarrow\xrightarrow{\alpha}\twoheadrightarrow q'$ for some q' with $(p',q') \in \mathcal{R}$.

Definition 3.6 (η-simulation)**.** A relation $\mathcal{R}$ is an *η-simulation* if, for all $(p,q) \in \mathcal{R}$, a step $p \xrightarrow{\alpha} p'$ implies (1) $\alpha = \tau$ and $(p',q) \in \mathcal{R}$, or (2) $q \twoheadrightarrow q' \xrightarrow{\alpha}\twoheadrightarrow q''$ for some q', q'' with $(p,q') \in \mathcal{R}$ and $(p',q'') \in \mathcal{R}$.

We will prove that the coordinates $e_{1\mathrm{S}} = (\infty,0,\infty,0,0,\infty,0,0)$ and $e_{\eta\mathrm{S}} = (\infty,\infty,\infty,0,0,\infty,0,0)$ of Definition 3.4 exactly define notions that match the operational definitions. The grammars induced by $e_{1\mathrm{S}}$ and $e_{\eta\mathrm{S}}$ to define $\preceq_{1\mathrm{S}}$ and $\preceq_{\eta\mathrm{S}}$ through $\mathcal{O}_{1\mathrm{S}}$ and $\mathcal{O}_{\eta\mathrm{S}}$ read:

$$\begin{array}{llcll}
\mathcal{O}_{1\mathrm{S}}: & \varphi_{1\mathrm{S}} & ::= & \langle\varepsilon\rangle\chi_{1\mathrm{S}} & \mid \ \mathsf{T} \\
& \chi_{1\mathrm{S}} & ::= & \langle a\rangle\varphi_{1\mathrm{S}} & \mid \ \bigwedge\{\psi_{1\mathrm{S}}, \psi_{1\mathrm{S}}, \ldots\} \\
& \psi_{1\mathrm{S}} & ::= & \langle\varepsilon\rangle\chi_{1\mathrm{S}} & \\
\mathcal{O}_{\eta\mathrm{S}}: & \varphi_{\eta\mathrm{S}} & ::= & \langle\varepsilon\rangle\chi_{\eta\mathrm{S}} & \mid \ \mathsf{T} \\
& \chi_{\eta\mathrm{S}} & ::= & \langle a\rangle\varphi_{\eta\mathrm{S}} & \mid \ \bigwedge\{\psi_{\eta\mathrm{S}}, \psi_{\eta\mathrm{S}}, \ldots\} \ \mid \ \bigwedge\{(\alpha)\varphi_{\eta\mathrm{S}}, \psi_{\eta\mathrm{S}}, \psi_{\eta\mathrm{S}}, \ldots\} \\
& \psi_{\eta\mathrm{S}} & ::= & \langle\varepsilon\rangle\chi_{\eta\mathrm{S}} &
\end{array}$$

We now turn to proving the characterizations correct. As there is considerable overlap, we give a single proof for both characterizations and refer to the notion of observability as N when we want to cover both cases.

Lemma 3.2. *The following characterizations hold:*

- *p_0 is weakly simulated by q_0, $p_0 \preceq_{1\mathrm{S}} q_0$, precisely if there is a weak simulation $\mathcal{R}_{1\mathrm{S}}$ with $(p_0,q_0) \in \mathcal{R}_{1\mathrm{S}}$.*
- *p_0 is η-simulated by q_0, $p_0 \preceq_{\eta\mathrm{S}} q_0$, precisely if there is an η-simulation $\mathcal{R}_{\eta\mathrm{S}}$ with $(p_0,q_0) \in \mathcal{R}_{\eta\mathrm{S}}$.*[3]

Proof. Let $N \in \{1\mathrm{S}, \eta\mathrm{S}\}$ be a notion of observability. According to Definitions 3.1 and 3.4, $p_0 \preceq_N q_0$ iff no $\varphi \in \mathcal{O}_N$ distinguishes p_0 from q_0. We prove that this is the case if and only if there is an N-simulation $\mathcal{R}_N$ (according to Definitions 3.5 and 3.6) with $(p_0,q_0) \in \mathcal{R}_N$.

[3] theorem `Eta_Bisimilarity.lts_tau.eta_sim_coordinate`

Assume $p_0 \preceq_{1S} q_0$. We need to find a weak simulation $\mathcal{R}_{1S}$ with $(p_0, q_0) \in \mathcal{R}_{1S}$: Consider $\mathcal{R}_{1S} := \{(p,q) \mid \forall \varphi \in \mathcal{O}_{1S}.\, p \in \llbracket \varphi \rrbracket \implies q \in \llbracket \varphi \rrbracket\}$. Because $p_0 \preceq_{1S} q_0$, we have $(p_0, q_0) \in \mathcal{R}_{1S}$. We will show $\mathcal{R}_{1S}$ to satisfy the operational definition of weak simulation (Definition 3.5).

We prove weak simulation on $\xrightarrow{a}$ by contradiction: Assume $p \xrightarrow{a} p'$ and $(p,q) \in \mathcal{R}_{1S}$, but for all q' with $q \twoheadrightarrow\xrightarrow{a}\twoheadrightarrow q'$, we have $(p', q') \notin \mathcal{R}_{1S}$. For each such q', there is a distinguishing formula $\langle\varepsilon\rangle\chi \in \mathcal{O}_{1S}$ with $p' \in \llbracket\langle\varepsilon\rangle\chi\rrbracket$ and $q' \notin \llbracket\langle\varepsilon\rangle\chi\rrbracket$. Set $\psi_{q'} := \langle\varepsilon\rangle\chi$. This $\psi_{q'}$ is a correct positive conjunct (of shape ψ in Definition 2.5). Then, $\langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\bigwedge\{\psi_{q'} \mid q \twoheadrightarrow\xrightarrow{a}\twoheadrightarrow q'\} \in \mathcal{O}_{1S}$ distinguishes p from q, contradicting $(p,q) \in \mathcal{R}_{1S}$.

We prove weak simulation on $\xrightarrow{\tau}$ by contradiction: Assume $p \xrightarrow{\tau} p'$ and $(p,q) \in \mathcal{R}_{1S}$, but for all q' with $q \twoheadrightarrow q'$, we have $(p', q') \notin \mathcal{R}_{1S}$. (Note that this includes $(p', q) \notin \mathcal{R}_{1S}$, as $q \twoheadrightarrow q$.) Similarly to the previous case, we find a $\psi_{q'}$ for each such q'. Then, $\langle\varepsilon\rangle\bigwedge\{\psi_{q'} \mid q \twoheadrightarrow q'\} \in \mathcal{O}_{1S}$ distinguishes p from q, contradicting $(p,q) \in \mathcal{R}_{1S}$.

Assume $p_0 \preceq_{\eta S} q_0$. We need to find an η-simulation $\mathcal{R}_{\eta S}$ with $(p_0, q_0) \in \mathcal{R}_{\eta S}$: Consider $\mathcal{R}_{\eta S} := \{(p,q) \mid \forall \varphi \in \mathcal{O}_{\eta S}.\, p \in \llbracket \varphi \rrbracket \implies q \in \llbracket \varphi \rrbracket\}$. Because $p_0 \preceq_{\eta S} q_0$, we have $(p_0, q_0) \in \mathcal{R}_{\eta S}$. We will show $\mathcal{R}_{\eta S}$ to satisfy the operational definition of η-simulation according to Definition 3.6 (by contradiction).

Assume $p \xrightarrow{\alpha} p'$ and $(p,q) \in \mathcal{R}_{\eta S}$, but for all q' and q'' with $q \twoheadrightarrow q' \xrightarrow{(\alpha)}\twoheadrightarrow q''$, $(p, q') \notin \mathcal{R}_{\eta S}$ or $(p', q'') \notin \mathcal{R}_{\eta S}$. Let

$$Q_\varepsilon := \{q' \mid q \twoheadrightarrow q' \wedge (p, q') \notin \mathcal{R}_{\eta S}\}$$
$$Q_\alpha := \{q' \mid q \twoheadrightarrow q' \wedge (p, q') \in \mathcal{R}_{\eta S} \wedge \forall q''.\, q' \xrightarrow{(\alpha)}\twoheadrightarrow q'' \implies (p', q'') \notin \mathcal{R}_{\eta S}\}.$$

We then define:

- For $q' \in Q_\varepsilon$, let $\varphi \in \mathcal{O}_{\eta S}$ be a formula that distinguishes p from q'. (Such a formula exists by the definition of $\mathcal{R}_{\eta S}$ and Q_ε.) Note that φ always has the form $\langle\varepsilon\rangle\chi$. We define $\psi_{\varepsilon,q'} := \langle\varepsilon\rangle\chi$.
- For $q' \in Q_\alpha$ and q'' with $q' \xrightarrow{(\alpha)}\twoheadrightarrow q''$, let $\varphi \in \mathcal{O}_{\eta S}$ be a formula that distinguishes p' from q''. We define $\psi_{\alpha,q',q''} := \varphi$.

Now consider $\varphi_\eta = \langle\varepsilon\rangle\bigwedge(\{(\alpha)\langle\varepsilon\rangle\bigwedge\{\psi_{\alpha,q',q''} \mid q' \in Q_\alpha, q' \xrightarrow{(\alpha)}\twoheadrightarrow q''\}\} \cup \{\psi_{\varepsilon,q'} \mid q' \in Q_\varepsilon\})$. This formula combines all the distinctions and holds for p because we know of the $p \xrightarrow{\alpha} p'$ transition. It cannot hold for q because all paths from q must either pass through Q_ε or Q_α. So it distinguishes p from q, contradicting $(p,q) \in \mathcal{R}_{\eta S}$.

Assume there is an N-simulation $\mathcal{R}_N$ with $(p_0, q_0) \in \mathcal{R}_N$. This means $p_0 \precsim_N q_0$, where $\precsim_N$ is the greatest N-simulation (i.e., the union of all N-simulations, which is referred to as "the weak/η-simulation preorder" in other publications). **We need to prove $p_0 \preceq_N q_0$:** We do so by proving that $p \precsim_N q$ implies $p \preceq_N q$ (i.e. $p \in \llbracket\varphi\rrbracket \implies q \in \llbracket\varphi\rrbracket$, for $\varphi \in \mathcal{O}_N$) for all processes p and q by mutual induction over the top-level formulas φ and (necessarily positive) conjuncts ψ.

Assume $p \precsim_N q$ and $p \in \llbracket\varphi\rrbracket$. The case $\varphi = \mathsf{T}$ is trivial. So we consider $\varphi = \langle\varepsilon\rangle\chi$. **We need to prove $q \in \llbracket\langle\varepsilon\rangle\chi\rrbracket$:** $p \in \llbracket\langle\varepsilon\rangle\chi\rrbracket$ implies there is $p \twoheadrightarrow p'$ such that $p' \in \llbracket\chi\rrbracket_\varepsilon$. The simulation property ensures a q' with $q \twoheadrightarrow q'$ such that still $p' \precsim_N q'$ (a detailed proof would use induction over the length of $p \twoheadrightarrow p'$). To prove $q \in \llbracket\langle\varepsilon\rangle\chi\rrbracket$, we will establish that there is $q'' \in \llbracket\chi\rrbracket_\varepsilon$ with $q \twoheadrightarrow q' \twoheadrightarrow q''$ by considering the cases for χ:

- Case $\chi = \langle a\rangle\varphi'$. Because $p' \in \llbracket\chi\rrbracket_\varepsilon$, this implies there is p'' with $p' \xrightarrow{a} p''$ and $p'' \in \llbracket\varphi'\rrbracket$. Due to η- or weak simulation, there are q'', q''' with $q \twoheadrightarrow q' \twoheadrightarrow q'' \xrightarrow{a}\twoheadrightarrow q'''$ and $p'' \precsim_N q'''$. Instantiating the induction hypothesis on subformula φ', this implies $q''' \in \llbracket\varphi'\rrbracket$. With the HML semantics, $q'' \in \llbracket\langle a\rangle\langle\varepsilon\rangle\varphi'\rrbracket_\varepsilon = \llbracket\langle a\rangle\varphi'\rrbracket_\varepsilon$, because φ' starts with $\langle\varepsilon\rangle$ or equals T.

- Case $\chi = \bigwedge\Psi$. We know that each $\psi \in \Psi$ must be true for p'. If we can choose an appropriate q'' with $q \twoheadrightarrow q' \twoheadrightarrow q''$ and $p' \precsim_N q''$, the induction hypothesis implies each ψ of the form $\langle\varepsilon\rangle\chi'$ to be true for q'' as well.
  If there is $\psi = (\alpha)\varphi' \in \Psi$ (which implies $N = \eta\mathrm{S}$), there are $p'' \in [\![\varphi']\!]$ and $p' \xrightarrow{(\alpha)} p''$. Then we can choose q'' to satisfy $q' \twoheadrightarrow q'' \xrightarrow{(\alpha)}\twoheadrightarrow q'''$ with $p' \precsim_{\eta\mathrm{S}} q''$ and $p'' \precsim_{\eta\mathrm{S}} q'''$, thanks to $\precsim_{\eta\mathrm{S}}$ being an η-simulation. By induction hypothesis, φ' must be true for q''' as well and thus $(\alpha)\varphi'$ holds for this q''.
  If no such conjunct is present, we just take $q'' = q'$.
  In every case, we have found a $q'' \in [\![\bigwedge\Psi]\!]_\varepsilon$.

Assume $p \precsim_N q$ and $p \in [\![\langle\varepsilon\rangle\chi]\!]_\wedge$. We need to prove $q \in [\![\langle\varepsilon\rangle\chi]\!]_\wedge$: This is exactly the same argument as in the top-level case $\varphi = \langle\varepsilon\rangle\chi$. □

Remark 3.3. The second part of above proof relies on structural induction on $\mathsf{HML}_{\mathrm{srbb}}$ formulas, and so will many proofs to come. Our grammar in Definition 2.5 and the expr-bounded subgrammars allow infinitary formulas. But as they are inductive objects, they only permit well-founded formulas. (In particular, there are no infinite trace formulas like "$\langle a\rangle^\omega$".) Therefore structural induction is valid.

3.3. Abstractions of Bisimilarity

Our weak spectrum of Definition 3.4 features a total of eight notions that abstract strong bisimilarity: the whole part north of (and including) contrasimilarity and stable bisimilarity.

We will first prove the coincidence of operational and coordinate characterization for the upper diamonds around (stability-respecting) η/branching/weak/delay bisimilarity, which subsumes Lemma 2.1.

Afterwards, we will turn our attention to the two lower antennas: contrasimilarity and stable bisimilarity. In general, there is much overlap of ideas between the proofs, and most of the core tricks have already been used for the simulations of the previous subsection.

Definition 3.7 (Delay simulation)**.** A relation $\mathcal{R}$ is a *delay simulation* if, for all $(p, q) \in \mathcal{R}$, a step $p \xrightarrow{\alpha} p'$ implies (1) $\alpha = \tau$ and $(p', q) \in \mathcal{R}$, or (2) $q \twoheadrightarrow q' \xrightarrow{\alpha} q''$ for some q', q'' with $(p', q'') \in \mathcal{R}$.

Lemma 3.3. *The following characterizations hold:*

- *p_0 and q_0 are weakly bisimilar, $p_0 \sim_{\mathrm{B}} q_0$, precisely if there is a symmetric weak simulation $\mathcal{R}_{\mathrm{B}}$ with $(p_0, q_0) \in \mathcal{R}_{\mathrm{B}}$.*
- *p_0 and q_0 are delay-bisimilar, $p_0 \sim_{\mathrm{DB}} q_0$, (respectively, stability-respecting delay-bisimilar, $p_0 \sim_{\mathrm{DB^{sr}}} q_0$), precisely if there is a (stability-respecting) symmetric delay simulation $\mathcal{R}_{\mathrm{DB}}$ with $(p_0, q_0) \in \mathcal{R}_{\mathrm{DB}}$.*
- *p_0 and q_0 are η-bisimilar, $p_0 \sim_\eta q_0$, precisely if there is a symmetric η-simulation $\mathcal{R}_\eta$ with $(p_0, q_0) \in \mathcal{R}_\eta$.*[4]
- *p_0 and q_0 are branching-bisimilar, $p_0 \sim_{\mathrm{BB}} q_0$, (respectively, stability-respecting branching-bisimilar, $p_0 \sim_{\mathrm{BB^{sr}}} q_0$) precisely if there is a (stability-respecting) branching bisimulation $\mathcal{R}_{\mathrm{BB}}$ with $(p_0, q_0) \in \mathcal{R}_{\mathrm{BB}}$.*[5]

Proof. Let $N \in \{\mathrm{BB^{sr}}, \mathrm{BB}, \eta, \mathrm{DB^{sr}}, \mathrm{DB}, \mathrm{B}\}$ be a notion of observability. We prove that there is no formula $\varphi \in \mathcal{O}_N$ distinguishing p_0 from q_0 or distinguishing q_0 from p_0, if and only if there is an N-bisimulation $\mathcal{R}_N$ (according to Definitions 2.4, 3.5, 3.6, and 3.7) with $(p_0, q_0) \in \mathcal{R}_N$.

Assume $p_0 \sim_{\mathrm{B}} q_0$. We need to find a symmetric weak simulation $\mathcal{R}_{\mathrm{B}}$ with $(p_0, q_0) \in \mathcal{R}_{\mathrm{B}}$: Consider $\mathcal{R}_{\mathrm{B}} \coloneqq \{(p, q) \mid \forall\varphi \in \mathcal{O}_{\mathrm{B}}.\, p \in [\![\varphi]\!] \iff q \in [\![\varphi]\!]\}$. Obviously $(p_0, q_0) \in \mathcal{R}_{\mathrm{B}}$. Also, the relation is symmetric. We will show $\mathcal{R}_{\mathrm{B}}$ to satisfy the operational definition of weak simulation.

[4] theorem `Eta_Bisimilarity.lts_tau.eta_bisim_coordinate`
[5] theorem `Branching_Bisimilarity.lts_tau.sr_branching_bisim_coordinate`

Assume $(p,q) \in \mathcal{R}_\mathrm{B}$ **and** $p \xrightarrow{a} p'$**. We need to find** q' **with (1a)** $q \twoheadrightarrow\xrightarrow{a}\twoheadrightarrow q'$ **and (2a)** $(p',q') \in \mathcal{R}_\mathrm{B}$**.** Assume to the contrary every q' satisfying (1a) has $(p',q') \notin \mathcal{R}_\mathrm{B}$. For each such q', there is:

- ...either a distinguishing formula $\langle\varepsilon\rangle\chi \in \mathcal{O}_\mathrm{B}$ with $p' \in [\![\langle\varepsilon\rangle\chi]\!]$ and $q' \notin [\![\langle\varepsilon\rangle\chi]\!]$. Set $\psi_{q'} := \langle\varepsilon\rangle\chi$.
- ...or a distinguishing formula $\langle\varepsilon\rangle\chi \in \mathcal{O}_\mathrm{B}$ with $q' \in [\![\langle\varepsilon\rangle\chi]\!]$ and $p \notin [\![\langle\varepsilon\rangle\chi]\!]$. Set $\psi_{q'} := \neg\langle\varepsilon\rangle\chi$.

In any case, $\psi_{q'}$ is a correct conjunct (of shape ψ in Definition 2.5). Then, $\langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\bigwedge\{\psi_{q'} \mid q \twoheadrightarrow\xrightarrow{a}\twoheadrightarrow q'\} \in \mathcal{O}_\mathrm{B}$ distinguishes p from q, contradicting $(p,q) \in \mathcal{R}_\mathrm{B}$.

Assume $(p,q) \in \mathcal{R}_\mathrm{B}$ **and** $p \xrightarrow{\tau} p'$**. We need to find** q' **with (1**τ**)** $q \twoheadrightarrow q'$ **and (2**τ**)** $(p',q') \in \mathcal{R}_\mathrm{B}$**.** Assume to the contrary every q' satisfying (1τ) has $(p',q') \notin \mathcal{R}_\mathrm{B}$. (Note that this includes $(p',q) \notin \mathcal{R}_\mathrm{B}$, as $q \twoheadrightarrow q$.) Similarly to the previous case, we construct $\psi_{q'}$ for every such q'. Then, $\langle\varepsilon\rangle\bigwedge\{\psi_{q'} \mid q \twoheadrightarrow q'\} \in \mathcal{O}_\mathrm{B}$ distinguishes p from q, contradicting $(p,q) \in \mathcal{R}_\mathrm{B}$.

Assume $p_0 \sim_\mathrm{DB} q_0$**. We need to find a symmetric delay simulation** $\mathcal{R}_\mathrm{DB}$ **with** $(p_0,q_0) \in \mathcal{R}_\mathrm{DB}$**:** Consider $\mathcal{R}_\mathrm{DB} := \{(p,q) \mid \forall\varphi \in \mathcal{O}_\mathrm{DB}.\, p \in [\![\varphi]\!] \iff q \in [\![\varphi]\!]\}$. We will show $\mathcal{R}_\mathrm{DB}$ to satisfy the operational definition of delay simulation.

Assume $(p,q) \in \mathcal{R}_\mathrm{DB}$ **and** $p \xrightarrow{a} p'$**. We need to find** q' **with (1a)** $q \twoheadrightarrow\xrightarrow{a} q'$ **and (2a)** $(p',q') \in \mathcal{R}_\mathrm{DB}$**:** Assume the contrary. So, for each q' satisfying (1a), there is:

- ...either a distinguishing formula $\varphi \in \mathcal{O}_\mathrm{DB}$ with $p' \in [\![\varphi]\!]$ and $q' \notin [\![\varphi]\!]$. If φ is some immediate conjunction $\bigwedge\Psi$, choose $\psi_{q'} \in \Psi$ such that $q' \notin [\![\psi_{q'}]\!]_\wedge$ (at least one such element of Ψ must exist). Otherwise, set $\psi_{q'} := \varphi$.
- ...or a distinguishing formula $\varphi \in \mathcal{O}_\mathrm{DB}$ with $q' \in [\![\varphi]\!]$ and $p' \notin [\![\varphi]\!]$. If φ is some immediate conjunction $\bigwedge\Psi$, choose $\widehat{\psi} \in \Psi$ such that $p \notin [\![\widehat{\psi}]\!]_\wedge$ (at least one such element of Ψ must exist) and let $\psi_{q'}$ be its negation. Otherwise, set $\psi_{q'} := \neg\varphi$.

In any case, $\psi_{q'}$ is a correct conjunct (of shape ψ in Definition 2.5). Then, $\langle\varepsilon\rangle\langle a\rangle\bigwedge\{\psi_{q'} \mid q \twoheadrightarrow\xrightarrow{a} q'\} \in \mathcal{O}_\mathrm{DB}$ distinguishes p from q, contradicting $(p,q) \in \mathcal{R}_\mathrm{DB}$.

Assume $p_0 \sim_{\mathrm{DB^{sr}}} q_0$**. We need to find a stability-respecting symmetric delay simulation** $\mathcal{R}_{\mathrm{DB^{sr}}}$ **with** $(p_0,q_0) \in \mathcal{R}_{\mathrm{DB^{sr}}}$**:** Consider $\mathcal{R}_{\mathrm{DB^{sr}}} := \{(p,q) \mid \forall\varphi \in \mathcal{O}_{\mathrm{DB^{sr}}}.\, p \in [\![\varphi]\!] \iff q \in [\![\varphi]\!]\}$. We will show $\mathcal{R}_{\mathrm{DB^{sr}}}$ to satisfy the operational definition of stability-respecting delay simulation.

Assume $(p,q) \in \mathcal{R}_{\mathrm{DB^{sr}}}$ **and** $p \xrightarrow{a} p'$**. We need to find** q' **with (1a)** $q \twoheadrightarrow\xrightarrow{a} q'$ **and (2a)** $(p',q') \in \mathcal{R}_{\mathrm{DB^{sr}}}$**:** This follows the same argumentation as the previous case ($p_0 \sim_\mathrm{DB} q_0$).

Assume $(p,q) \in \mathcal{R}_{\mathrm{DB^{sr}}}$ **and** $p \stackrel{\tau}{\not\to}$**. We need to find** q' **with (1s)** $q \twoheadrightarrow q' \stackrel{\tau}{\not\to}$ **and (2s)** $(p,q') \in \mathcal{R}_{\mathrm{DB^{sr}}}$**:** Assume the contrary. So, for each q' satisfying (1s), there is:

- ...either a distinguishing formula $\varphi \in \mathcal{O}_{\mathrm{DB^{sr}}}$ with $p \in [\![\varphi]\!]$ and $q' \notin [\![\varphi]\!]$. If φ is some immediate conjunction $\bigwedge\Psi$, choose $\psi_{q'} \in \Psi$ such that $q' \notin [\![\psi_{q'}]\!]_\wedge$ (at least one such element of Ψ must exist). Otherwise, set $\psi_{q'} := \varphi$.
- ...or a distinguishing formula $\varphi \in \mathcal{O}_{\mathrm{DB^{sr}}}$ with $q' \in [\![\varphi]\!]$ and $p \notin [\![\varphi]\!]$. If φ is some immediate conjunction $\bigwedge\Psi$, choose $\widehat{\psi} \in \Psi$ such that $p \notin [\![\widehat{\psi}]\!]_\wedge$ (at least one such element of Ψ must exist) and let $\psi_{q'}$ be its negation. Otherwise, set $\psi_{q'} := \neg\varphi$.

In any case, $\psi_{q'}$ is a correct conjunct (of shape ψ in Definition 2.5). Then, $\langle\varepsilon\rangle\bigwedge(\{\neg\langle\tau\rangle\mathsf{T}\} \cup \{\psi_{q'} \mid q \twoheadrightarrow q' \stackrel{\tau}{\not\to}\}) \in \mathcal{O}_{\mathrm{DB^{sr}}}$ distinguishes p from q, contradicting $(p,q) \in \mathcal{R}_{\mathrm{DB^{sr}}}$.

Assume $p_0 \sim_\eta q_0$**. We need to find a symmetric** η**-simulation** $\mathcal{R}_\eta$ **with** $(p_0,q_0) \in \mathcal{R}_\eta$**:** Consider $\mathcal{R}_\eta := \{(p,q) \mid \forall\varphi \in \mathcal{O}_\eta.\, p \in [\![\varphi]\!] \iff q \in [\![\varphi]\!]\}$. We will show $\mathcal{R}_\eta$ to satisfy the operational definition of η-simulation.

Assume $(p,q) \in \mathcal{R}_\eta$ **and** $p \xrightarrow{\alpha} p'$**. We need to find** q' **and** q'' **with (1**η**)** $q \twoheadrightarrow q' \xrightarrow{(\alpha)}\twoheadrightarrow q''$ **and (2**η**)** $(p,q') \in \mathcal{R}_\eta$ **and (3**η**)** $(p',q'') \in \mathcal{R}_\eta$**:** If $\alpha = \tau$ and $(p',q) \in \mathcal{R}_\eta$, we can choose $q' = q'' = q$ and are finished.

Otherwise, assume the contrary. So, let Q_ε be the set of relevant q' satisfying (1η) but not (2η) and Q_α the set of q' satisfying (1η) and (2η), but for which every q'' violates (3η):

$$Q_\varepsilon = \{q' \mid q \twoheadrightarrow q' \wedge (p, q') \notin \mathcal{R}_\eta\}$$
$$Q_\alpha = \{q' \mid q \twoheadrightarrow q' \wedge (p, q') \in \mathcal{R}_\eta \wedge \forall q''. q' \xrightarrow{(\alpha)}\!\!\twoheadrightarrow q'' \implies (p', q'') \notin \mathcal{R}_\eta\}$$

We then define:

- For $q' \in Q_\varepsilon$, let $\varphi \in \mathcal{O}_\eta$ be a formula that distinguishes p from q', or a formula that distinguishes q' from p. We define $\psi_{\varepsilon,q'}$ as above, where we prove respect of stability.
- For $q' \in Q_\alpha$ and q'' with $q' \xrightarrow{(\alpha)}\!\!\twoheadrightarrow q''$, let $\varphi \in \mathcal{O}_\eta$ be a formula that distinguishes p' from q'', or a formula that distinguishes q'' from p'. We define $\psi_{\alpha,q',q''}$ similarly to where we prove respect of stability.

Now consider $\varphi_\eta = \langle\varepsilon\rangle \bigwedge (\{(a)\langle\varepsilon\rangle \bigwedge \{\psi_{\alpha,q',q''} \mid q' \in Q_a, q' \xrightarrow{(\alpha)}\!\!\twoheadrightarrow q''\}\} \cup \{\psi_{\varepsilon,q'} \mid q' \in Q_\varepsilon\})$. This formula combines all the distinctions and holds for p because of $p \xrightarrow{a} p'$. It cannot hold for q because all paths from q must either pass through Q_ε or Q_α. So it distinguishes p from q, contradicting $(p, q) \in \mathcal{R}_\eta$.

Assume $p_0 \sim_{\mathrm{BB}} q_0$. We need to find a branching bisimulation $\mathcal{R}_{\mathrm{BB}}$ with $(p_0, q_0) \in \mathcal{R}_{\mathrm{BB}}$: Consider $\mathcal{R}_{\mathrm{BB}} := \{(p, q) \mid \forall \varphi \in \mathcal{O}_{\mathrm{BB}}. p \in [\![\varphi]\!] \iff q \in [\![\varphi]\!]\}$. We will show $\mathcal{R}_{\mathrm{BB}}$ to satisfy the operational definition of branching bisimulation.

Assume $(p, q) \in \mathcal{R}_{\mathrm{BB}}$ and $p \xrightarrow{\alpha} p'$. We need to find q' and q'' with (1α) $q \twoheadrightarrow q' \xrightarrow{(\alpha)} q''$ and (2α) $(p, q') \in \mathcal{R}_{\mathrm{BB}}$ and (3α) $(p', q'') \in \mathcal{R}_{\mathrm{BB}}$: If $\alpha = \tau$ and $(p', q) \in \mathcal{R}_{\mathrm{BB}}$, we can choose $q' = q'' = q$ and are finished.

Otherwise, assume the contrary. Similar to the previous case ($p_0 \sim_\eta q_0$), we let:

$$Q_\varepsilon = \{q' \mid q \twoheadrightarrow q' \wedge (p, q') \notin \mathcal{R}_{\mathrm{BB}}\}$$
$$Q_\alpha = \{q' \mid q \twoheadrightarrow q' \wedge (p, q') \in \mathcal{R}_{\mathrm{BB}} \wedge \forall q''. q' \xrightarrow{(\alpha)} q'' \implies (p', q'') \notin \mathcal{R}_{\mathrm{BB}}\}$$

and continue as in the previous case.

Assume $p_0 \sim_{\mathrm{BB^{sr}}} q_0$. We need to find a stability-respecting branching bisimulation $\mathcal{R}_{\mathrm{BB^{sr}}}$ with $(p_0, q_0) \in \mathcal{R}_{\mathrm{BB^{sr}}}$: Consider $\mathcal{R}_{\mathrm{BB^{sr}}} := \{(p, q) \mid \forall \varphi \in \mathcal{O}_{\mathrm{BB^{sr}}}. p \in [\![\varphi]\!] \iff q \in [\![\varphi]\!]\}$. We will show $\mathcal{R}_{\mathrm{BB^{sr}}}$ to satisfy the operational definition of stability-respecting branching bisimulation.

Assume $(p, q) \in \mathcal{R}_{\mathrm{BB^{sr}}}$ and $p \xrightarrow{\alpha} p'$. We need to find q' and q'' with (1α) $q \twoheadrightarrow q' \xrightarrow{(\alpha)} q''$ and (2α) $(p, q') \in \mathcal{R}_{\mathrm{BB^{sr}}}$ and (3α) $(p', q'') \in \mathcal{R}_{\mathrm{BB^{sr}}}$: This follows the same argumentation as the previous case ($p_0 \sim_{\mathrm{BB}} q_0$).

Assume $(p, q) \in \mathcal{R}_{\mathrm{BB^{sr}}}$ and $p \not\xrightarrow{\tau}$. We need to find q' with (1s) $q \twoheadrightarrow q' \not\xrightarrow{\tau}$ and (2s) $(p, q') \in \mathcal{R}_{\mathrm{DB^{sr}}}$: This follows the same argumentation as the case of stability-respecting delay simulation ($p_0 \sim_{\mathrm{DB^{sr}}} q_0$).

For the other direction, assume there is a symmetric N-simulation $\mathcal{R}_N$ with $(p_0, q_0) \in \mathcal{R}_N$. This means $p_0 \approx_N q_0$, where $\approx_N$ is the greatest symmetric N-simulation (according to Definitions 2.4, 3.5, 3.6, and 3.7). **We need to prove $p_0 \sim_N q_0$:** We do so by proving that $p \approx_N q$ implies $p \preceq_N q$ (i.e. $p \in [\![\varphi]\!] \implies q \in [\![\varphi]\!]$, for $\varphi \in \mathcal{O}_N$) for all processes p and q by mutual induction over the top-level formulas φ and conjuncts ψ. Because $\approx_N$ is symmetric, this will prove $p_0 \in [\![\varphi]\!] \iff q_0 \in [\![\varphi]\!]$ for all $\varphi \in \mathcal{O}_N$, so we are then finished.

Assume $p \approx_N q$ and $p \in [\![\langle\varepsilon\rangle\chi]\!]$. We need to prove $q \in [\![\langle\varepsilon\rangle\chi]\!]$: $p \in [\![\langle\varepsilon\rangle\chi]\!]$ implies there is $p \twoheadrightarrow p'$ such that $p' \in [\![\chi]\!]_\varepsilon$. To prove $q \in [\![\langle\varepsilon\rangle\chi]\!]$, we will establish that there is $q' \in [\![\chi]\!]_\varepsilon$ with $q \twoheadrightarrow q'$ by considering the cases for χ:

- Case $\chi = \langle a \rangle \langle \varepsilon \rangle \chi'$. This implies there is p'' with $p' \xrightarrow{a} p''$ and $p'' \in [\![\langle \varepsilon \rangle \chi']\!]$. Due to branching, delay, η-, or weak simulation, there are q', q'' with $q \twoheadrightarrow q' \xrightarrow{a}\twoheadrightarrow q''$ and $p'' \approx_N q''$. With the induction hypothesis, this implies $q'' \in [\![\langle \varepsilon \rangle \chi']\!]$, and due to the HML semantics, $q' \in [\![\langle a \rangle \langle \varepsilon \rangle \chi']\!]_\varepsilon$.
- Case $\chi = \langle a \rangle \bigwedge \Psi$. (This formula only appears if $N \in \{\mathrm{BB}^{\mathrm{sr}}, \mathrm{BB}, \mathrm{DB}^{\mathrm{sr}}, \mathrm{DB}\}$.) This implies there is p'' with $p' \xrightarrow{a} p''$ and $p'' \in [\![\bigwedge \Psi]\!]$. Due to branching or delay simulation, there are q', q'' with $q \twoheadrightarrow q' \xrightarrow{a} q''$ and $p'' \approx_N q''$. With the induction hypothesis, this implies $q'' \in [\![\bigwedge \Psi]\!]$, and due to the HML semantics, $q' \in [\![\langle a \rangle \bigwedge \Psi]\!]_\varepsilon$.
- Case $\chi = \bigwedge \Psi$. We know that each $\psi \in \Psi$ must be true for p'. If we can choose an appropriate q' with $q \twoheadrightarrow q'$ and $p' \approx_N q'$, the induction hypothesis implies each ψ of the form $\neg \langle \varepsilon \rangle \chi'$ or $\langle \varepsilon \rangle \chi'$ to be true for q' as well.
  If there is $\psi = \neg \langle \tau \rangle \mathsf{T} \in \Psi$ (which implies $N \in \{\mathrm{BB}^{\mathrm{sr}}, \mathrm{DB}^{\mathrm{sr}}\}$), its truth ensures $p' \not\xrightarrow{\tau}$. We can choose q' to satisfy $q \twoheadrightarrow q' \not\xrightarrow{\tau}$ and $p' \approx_N q'$, thanks to $\approx_N$ respecting stability. q' satisfies $\neg \langle \tau \rangle \mathsf{T}$.
  If, otherwise, there is $\psi = (\alpha)\varphi' \in \Psi$ (which implies $N \in \{\mathrm{BB}^{\mathrm{sr}}, \mathrm{BB}, \eta\}$), there are $p'' \in [\![\varphi']\!]$ and $p' \xrightarrow{(\alpha)} p''$. Then we can choose q' to satisfy $q \twoheadrightarrow q' \xrightarrow{(\alpha)}\twoheadrightarrow q''$ with $p' \approx_N q'$ and $p'' \approx_N q''$, thanks to $\approx_N$ being a branching or η-simulation. By induction hypothesis, φ' must be true for q'' as well and thus $(\alpha)\varphi'$ holds for this q'.
  If neither of the prior two conjuncts are present, we just take $q' = q$ if $p = p'$, or otherwise some q' with $q \twoheadrightarrow q'$ and $p' \approx_N q'$ that is implied by the simulation property of $\approx_N$ on $p \xrightarrow{\tau}{}^n p'$.
  In every case, we have found a $q' \in [\![\bigwedge \Psi]\!]$.

Assume $p \approx_N q$ **and** $p \in [\![\bigwedge \Psi]\!]$**.** (This formula only appears if $N \in \{\mathrm{BB}^{\mathrm{sr}}, \mathrm{BB}, \mathrm{DB}^{\mathrm{sr}}, \mathrm{DB}\}$.) **We need to prove** $q \in [\![\bigwedge \Psi]\!]$**:** We know that each $\psi \in \Psi$ must be true for p. By induction hypothesis, $p \approx_N q$ implies each ψ to be true for q as well. Thus $q \in [\![\bigwedge \Psi]\!]$.

Now turn to the conjunct formulas, denoted by ψ:

Assume $p \approx_N q$ **and** $p \in [\![\langle \varepsilon \rangle \chi]\!]_\wedge$**. We need to prove** $q \in [\![\langle \varepsilon \rangle \chi]\!]_\wedge$**:** The proof of the first case also addresses this case.

Assume $p \approx_N q$ **and** $p \in [\![\neg \langle \varepsilon \rangle \chi]\!]_\wedge$**. We need to prove** $q \in [\![\neg \langle \varepsilon \rangle \chi]\!]_\wedge$**:** Thanks to the proof of the previous case and symmetry of $\approx_N$, we know that if $\langle \varepsilon \rangle \chi$ were to be true for q, it would also need to be true for p. The case implies that $p \notin [\![\langle \varepsilon \rangle \chi]\!]$. By contraposition, $q \notin [\![\langle \varepsilon \rangle \chi]\!]$. This proves that $\neg \langle \varepsilon \rangle \chi$, in the context of a conjunction, does hold in q. □

For the weaker abstractions of bisimilarity, that is, contrasimilarity with $e_{\mathrm{C}} = (\infty, 0, \infty, 0, 0, 0, \infty, \infty)$ and stable bisimilarity $e_{\mathrm{SB}} = (\infty, 0, 0, \infty, 0, \infty, \infty, \infty)$, our coordinate characterization matches the modal logics used in Bisping and Montanari [21] (extended by some trivial T subterms). We can prove the adequacy of the characterizations similarly to the stronger abstractions.

Definition 3.4 defines contrasimulation and stable bisimulation preorders through the following formula languages, where the χ_{U}-parts do not add any distinguishing power:

$$
\begin{array}{llll}
\mathcal{O}_{\mathrm{C}}: & \varphi_{\mathrm{C}} & ::= & \langle \varepsilon \rangle \chi_{\mathrm{C}} \quad | \quad \mathsf{T} \\
& \chi_{\mathrm{C}} & ::= & \langle a \rangle \varphi_{\mathrm{C}} \quad | \quad \bigwedge \{\psi_{\mathrm{C}}, \psi_{\mathrm{C}}, \dots\} \\
& \psi_{\mathrm{C}} & ::= & \neg \langle \varepsilon \rangle \chi_{\mathrm{C}} \quad | \quad \langle \varepsilon \rangle \chi_{\mathrm{U}} \\
& \chi_{\mathrm{U}} & ::= & \bigwedge \{\langle \varepsilon \rangle \chi_{\mathrm{U}}, \dots, \neg \langle \varepsilon \rangle \chi_{\mathrm{U}}\}
\end{array}
$$

$$
\begin{array}{llll}
\mathcal{O}_{\text{SB}}: & \varphi_{\text{SB}} & ::= & \langle\varepsilon\rangle\chi_{\text{SB}} \quad | \quad \top \\
& \chi_{\text{SB}} & ::= & \langle a\rangle\varphi_{\text{SB}} \quad | \quad \bigwedge\{\neg\langle\tau\rangle\top, \psi_{\text{SB}}, \psi_{\text{SB}}, \ldots\} \\
& \psi_{\text{SB}} & ::= & \neg\langle\varepsilon\rangle\chi_{\text{SB}} \quad | \quad \langle\varepsilon\rangle\chi_{\text{SB}}
\end{array}
$$

The common characterization through contrasimulation relations (on words!) is as follows:

Definition 3.8 (Contrasimulation)**.** A *contrasimulation* is a relation $\mathcal{R}$ where, for all $(p,q) \in \mathcal{R}$ with $\vec{w} \in \Sigma^*$ and $p \overset{\vec{w}}{\twoheadrightarrow} p'$, there is a q' with $q \overset{\vec{w}}{\twoheadrightarrow} q'$ and $(q',p') \in \mathcal{R}$.

For the following proof, we introduce the abbreviation $\langle\vec{w}\rangle\varphi$, for $\varphi \in \mathcal{O}_\text{C}$, to mean:

- If $\vec{w}$ is the empty word λ, we set $\langle\vec{w}\rangle\varphi := \varphi$.
- If $\vec{w} = \vec{w}'\tau$, we set $\langle\vec{w}\rangle\varphi := \langle\vec{w}'\rangle\varphi$. (Note that φ always has the form $\langle\varepsilon\rangle\chi$, as immediate conjunctions are not allowed by $\mathcal{O}_\text{C}$.)
- If $\vec{w} = \vec{w}'a$, we set $\langle\vec{w}\rangle\varphi := \langle\vec{w}'\rangle\langle\varepsilon\rangle\langle a\rangle\varphi$.

Lemma 3.4. *p_0 is contrasimulation-preordered to q_0, $p_0 \preceq_\text{C} q_0$, precisely if there is a contrasimulation $\mathcal{R}_\text{C}$ with $(p_0,q_0) \in \mathcal{R}_\text{C}$.*

Proof. We prove that there is no formula $\varphi \in \mathcal{O}_\text{C}$ distinguishing p_0 from q_0, if and only if there is a contrasimulation $\mathcal{R}_\text{C}$ with $(p_0,q_0) \in \mathcal{R}_\text{C}$.

Assume no $\varphi \in \mathcal{O}_\text{C}$ distinguishes p_0 from q_0. We need to find a contrasimulation $\mathcal{R}_\text{C}$ with $(p_0,q_0) \in \mathcal{R}_\text{C}$. Consider $\mathcal{R}_\text{C} := \{(p,q) \mid \forall\varphi \in \mathcal{O}_\text{C}.\, p \in [\![\varphi]\!] \implies q \in [\![\varphi]\!]\}$. Due to the absence of distinction, $(p_0,q_0) \in \mathcal{R}_\text{C}$. We will show $\mathcal{R}_\text{C}$ to satisfy the operational definition of contrasimulation (Definition 3.8).

We need to prove contrasimulation on $\overset{\vec{w}}{\twoheadrightarrow}$ (by contradiction): Assume $p \overset{\vec{w}}{\twoheadrightarrow} p'$ and $(p,q) \in \mathcal{R}_\text{C}$, but for all q' with $q \overset{\vec{w}}{\twoheadrightarrow} q'$, we have $(q',p') \notin \mathcal{R}_\text{C}$. For each such q', there is a distinguishing formula $\varphi \in \mathcal{O}_\text{C}$ with $q' \in [\![\varphi]\!]$ and $p' \notin [\![\varphi]\!]$. Set $\psi_{q'} := \neg\varphi$. Formula $\psi_{q'}$ is a correct negative conjunct (of shape ψ in Definition 2.5). Then, $\langle\vec{w}\rangle\langle\varepsilon\rangle\bigwedge\{\psi_{q'} \mid q \overset{\vec{w}}{\twoheadrightarrow} q'\} \in \mathcal{O}_\text{C}$ distinguishes p from q, contradicting $(p,q) \in \mathcal{R}_\text{C}$.

For the other direction, assume there is a contrasimulation $\mathcal{R}$ with $(p_0,q_0) \in \mathcal{R}$. This means $p_0 \precsim_\text{C} q_0$, where $\precsim_\text{C}$ is the greatest contrasimulation. **We need to prove $p_0 \preceq_\text{C} q_0$:** We do so by proving that $p \precsim_\text{C} q$ implies $p \preceq_\text{C} q$ (i.e. $p \in [\![\varphi]\!] \implies q \in [\![\varphi]\!]$, for $\varphi \in \mathcal{O}_\text{C}$) for all processes p and q by induction over the structure of φ.

Assume $p \precsim_\text{C} q$ and $p \in [\![\varphi]\!]$. Note that φ always can be written in the form $\langle\vec{w}\rangle\langle\varepsilon\rangle\bigwedge\Psi$, for a suitable word $\vec{w}$ and set of negative or trivial conjuncts $\Psi = \{\neg\langle\varepsilon\rangle\chi', \ldots, \langle\varepsilon\rangle\chi_\text{U}, \ldots\}$. **We need to prove $q \in [\![\langle\vec{w}\rangle\langle\varepsilon\rangle\bigwedge\Psi]\!]$:** Then, $p \in [\![\langle\vec{w}\rangle\langle\varepsilon\rangle\bigwedge\Psi]\!]$ implies there is $p \overset{\vec{w}}{\twoheadrightarrow} p'$ such that $p' \in [\![\bigwedge\Psi]\!]_\varepsilon$. That means that for all conjuncts $\neg\langle\varepsilon\rangle\chi' \in \Psi$, we have $p' \notin [\![\langle\varepsilon\rangle\chi']\!]$. (The positive conjuncts $\langle\varepsilon\rangle\chi_\text{U} \in \Psi$ must be logically equivalent to $\top$. We can thus ignore them.)

Now let q' be the process with $q \overset{\vec{w}}{\twoheadrightarrow} q'$ and $q' \precsim_\text{C} p'$, which exists by the contrasimulation property. q' must satisfy $\bigwedge\Psi$ because otherwise, there is some conjunct $\neg\langle\varepsilon\rangle\chi' \in \Psi$ such that $q' \notin [\![\neg\langle\varepsilon\rangle\chi']\!]$. That means $q' \in [\![\langle\varepsilon\rangle\chi']\!]$, and therefore $p' \in [\![\langle\varepsilon\rangle\chi']\!]$ by induction hypothesis. But the latter contradicts $p' \notin [\![\langle\varepsilon\rangle\chi']\!]$. □

Definition 3.9 (Stable bisimulation)**.** A relation $\mathcal{R}$ is a *stable bisimulation* if, for all $(p,q) \in \mathcal{R}$ with $\vec{w} \in \Sigma^*$ and $p \overset{\vec{w}}{\twoheadrightarrow} p'$, there is a q' with $q \overset{\vec{w}}{\twoheadrightarrow} q'$, and in case $p' \overset{\tau}{\not\twoheadrightarrow}$, moreover $q' \overset{\tau}{\not\twoheadrightarrow}$ and $(p',q') \in \mathcal{R} \cap \mathcal{R}^{-1}$.

Lemma 3.5. *p_0 is stable bisimulation-preordered to q_0, $p_0 \preceq_\text{SB} q_0$, precisely if there is a stable bisimulation $\mathcal{R}_\text{SB}$ with $(p_0,q_0) \in \mathcal{R}_\text{SB}$.*

Proof. We prove that there is no formula $\varphi \in \mathcal{O}_{\text{SB}}$ distinguishing p_0 from q_0, if and only if there is a stable bisimulation $\mathcal{R}_{\text{SB}}$ with $(p_0, q_0) \in \mathcal{R}_{\text{SB}}$.

Assume no $\varphi \in \mathcal{O}_{\text{SB}}$ distinguishes p_0 from q_0. We need to find a stable bisimulation $\mathcal{R}_{\text{SB}}$ with $(p_0, q_0) \in \mathcal{R}_{\text{SB}}$: Consider $\mathcal{R}_{\text{SB}} := \{(p, q) \mid \forall \varphi \in \mathcal{O}_{\text{SB}}.\, p \in [\![\varphi]\!] \implies q \in [\![\varphi]\!]\}$. Due to the absence of distinction, $(p_0, q_0) \in \mathcal{R}_{\text{SB}}$. We will show $\mathcal{R}_{\text{SB}}$ to satisfy the operational definition of stable bisimulation (Definition 3.9).

We need to prove trace inclusion on $\overset{\vec{w}}{\twoheadrightarrow}$ (by contradiction): Assume $p \overset{\vec{w}}{\twoheadrightarrow} p'$ and $(p, q) \in \mathcal{R}_{\text{SB}}$, but there is no q' with $q \overset{\vec{w}}{\twoheadrightarrow} q'$. Then it is easy to see that $\langle \vec{w} \rangle \mathsf{T}$ is a formula that distinguishes p from q, contradicting $(p, q) \in \mathcal{R}_{\text{SB}}$. ($\mathsf{T} = \bigwedge \varnothing$ formally is an immediate conjunction but is in $\mathcal{O}_{\text{SB}}$, as $\mathsf{expr}(\mathsf{T}) = \mathbf{0}$.)

We need to prove respect of stability (after a word step $\overset{\vec{w}}{\twoheadrightarrow}$): Assume there is $(p, q) \in \mathcal{R}_{\text{SB}}$ with $p \overset{\vec{w}}{\twoheadrightarrow} p' \overset{\tau}{\not\rightarrow}$ but all q' with $q \overset{\vec{w}}{\twoheadrightarrow} q' \overset{\tau}{\not\rightarrow}$ have $(p', q') \notin \mathcal{R}_{\text{SB}} \cap \mathcal{R}_{\text{SB}}^{-1}$. For each such q' with $q \overset{\vec{w}}{\twoheadrightarrow} q' \overset{\tau}{\not\rightarrow}$, there is:

- ... either a distinguishing formula $\varphi \in \mathcal{O}_{\text{SB}}$ with $p' \in [\![\varphi]\!]$ and $q' \notin [\![\varphi]\!]$. Set $\psi_{q'} := \varphi$.
- ... or a distinguishing formula $\varphi \in \mathcal{O}_{SB}$ with $q' \in [\![\varphi]\!]$ and $p' \notin [\![\varphi]\!]$. Set $\psi_{q'} := \neg\varphi$, with the understanding that $\neg\neg\varphi$ is simplified to φ.

In any case, $\psi_{q'}$ is a correct conjunct (of shape ψ in Definition 2.5). Then,

$$\langle \vec{w} \rangle \langle \varepsilon \rangle \bigwedge \left(\{\neg\langle \tau \rangle \mathsf{T}\} \cup \{\psi_{q'} \mid q \overset{\vec{w}}{\twoheadrightarrow} q' \overset{\tau}{\not\rightarrow}\}\right) \in \mathcal{O}_{\text{SB}}$$

distinguishes p from q, contradicting $(p, q) \in \mathcal{R}_{\text{SB}}$.

For the other direction, assume there is a stable bisimulation relating p_0 and q_0. This means $p_0 \precsim_{\text{SB}} q_0$, where $\precsim_{\text{SB}}$ is the greatest stable bisimulation. **We need to prove $p_0 \preceq_{\text{SB}} q_0$:** We do so by proving that $p \precsim_{\text{SB}} q$ implies $p \preceq_{\text{SB}} q$ (i.e., $p \in [\![\varphi]\!] \implies q \in [\![\varphi]\!]$, for $\varphi \in \mathcal{O}_{\text{SB}}$) for all processes p and q by mutual induction over the structure of φ (and inner ψ, if p and q are stable).

Assume $p \precsim_{\text{SB}} q$ and $p \in [\![\langle \varepsilon \rangle \langle a \rangle \langle \varepsilon \rangle \chi']\!]$. We need to prove $q \in [\![\langle \varepsilon \rangle \langle a \rangle \langle \varepsilon \rangle \chi']\!]$: This implies there are $p \twoheadrightarrow p' \overset{a}{\rightarrow} p''$ and $p'' \in [\![\langle \varepsilon \rangle \chi']\!]$. Due to stable bisimulation, there are q', q'' with $q \twoheadrightarrow q' \overset{a}{\rightarrow} q''$ and $p'' \approx_N q''$. With the induction hypothesis, this implies $q'' \in [\![\langle \varepsilon \rangle \chi']\!]$, and the HML semantics implies the required property.

Assume $p \precsim_{\text{SB}} q$ and $p \in [\![\langle \varepsilon \rangle \bigwedge \{\neg\langle \tau \rangle \mathsf{T}\} \cup \Psi]\!]$. We need to prove $q \in [\![\langle \varepsilon \rangle \bigwedge \{\neg\langle \tau \rangle \mathsf{T}\} \cup \Psi]\!]$: This implies there is $p \twoheadrightarrow p'$ such that $p' \in [\![\bigwedge \{\neg\langle \tau \rangle \mathsf{T}\} \cup \Psi]\!]_\varepsilon$. We know that each $\psi \in \Psi$ must be true for p', and that there exists a q' with $p' \precsim_{\text{SB}} q'$ and $q \twoheadrightarrow q' \overset{\tau}{\not\rightarrow}$. The induction hypothesis implies that each $\psi \in \Psi$ of the form $\neg\langle \varepsilon \rangle \chi'$ or $\langle \varepsilon \rangle \chi'$ is true for q' as well, so $q' \in [\![\bigwedge \{\neg\langle \tau \rangle \mathsf{T}\} \cup \Psi]\!]_\varepsilon$, which implies the required property.

Now turn to the conjunct formulas, denoted by ψ. Note that we only need to consider stable states.

Assume $p \precsim_{\text{SB}} q$, $p \overset{\tau}{\not\rightarrow}$, $q \overset{\tau}{\not\rightarrow}$ and $p \in [\![\langle \varepsilon \rangle \chi]\!]_\wedge$. We need to prove $q \in [\![\langle \varepsilon \rangle \chi]\!]_\wedge$: The proofs of the two cases above also imply this case.

Assume $p \precsim_{\text{SB}} q$, $p \overset{\tau}{\not\rightarrow}$, $q \overset{\tau}{\not\rightarrow}$ and $p \in [\![\neg\langle \varepsilon \rangle \chi]\!]_\wedge$. We need to prove $q \in [\![\neg\langle \varepsilon \rangle \chi]\!]_\wedge$: Thanks to the proof of the previous case and symmetry of stable bisimulation on stable states, we know that if $\langle \varepsilon \rangle \chi$ were to be true for q, it would also need to be true for p. The assumption implies that $p \notin [\![\langle \varepsilon \rangle \chi]\!]_\wedge$. By contraposition, $q \notin [\![\langle \varepsilon \rangle \chi]\!]_\wedge$. This proves that $\neg\langle \varepsilon \rangle \chi$ does hold in q. □

(Unstable) readiness $e_{\mathrm{R}} = (\infty, 0, 1, 0, 0, 1, 1, 1)$.

$$\begin{array}{lcl} \varphi_{\mathrm{R}} & ::= & \langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{R}} \mid \langle\varepsilon\rangle\bigwedge\{\psi_{\mathrm{R}}, \psi_{\mathrm{R}}, \ldots\} \mid \mathsf{T} \\ \psi_{\mathrm{R}} & ::= & \neg\langle\varepsilon\rangle\langle a\rangle\mathsf{T} \mid \neg\langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\mathsf{T} \mid \langle\varepsilon\rangle\langle a\rangle\mathsf{T} \mid \langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\mathsf{T} \mid \langle\varepsilon\rangle\mathsf{T} \end{array}$$

Stable readiness $e_{\mathrm{R^s}} = (\infty, 0, 0, 1, 0, 1, 1, 1)$.

$$\varphi_{\mathrm{R^s}} \quad ::= \quad \langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{R^s}} \mid \langle\varepsilon\rangle\bigwedge\{\neg\langle\tau\rangle, \psi_{\mathrm{R}}, \psi_{\mathrm{R}}, \ldots\} \mid \mathsf{T} \mid \langle\varepsilon\rangle\mathsf{T}$$

(Unstable) impossible futures $e_{\mathrm{IF}} = (\infty, 0, 1, 0, 0, 0, \infty, 1)$.

$$\begin{array}{lcl} \varphi_{\mathrm{IF}} & ::= & \langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{IF}} \mid \langle\varepsilon\rangle\bigwedge\{\psi_{\mathrm{IF}}, \psi_{\mathrm{IF}}, \ldots\} \mid \mathsf{T} \\ \psi_{\mathrm{IF}} & ::= & \neg\langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{T}} \mid \neg\langle\varepsilon\rangle\mathsf{T} \mid \langle\varepsilon\rangle\mathsf{T} \end{array}$$

Stable impossible futures $e_{\mathrm{IF^s}} = (\infty, 0, 0, 1, 0, 0, \infty, 1)$.

$$\varphi_{\mathrm{IF^s}} \quad ::= \quad \langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{IF^s}} \mid \langle\varepsilon\rangle\bigwedge\{\neg\langle\tau\rangle, \psi_{\mathrm{IF}}, \psi_{\mathrm{IF}}, \ldots\} \mid \mathsf{T} \mid \langle\varepsilon\rangle\mathsf{T}$$

(Unstable) possible futures $e_{\mathrm{PF}} = (\infty, 0, 1, 0, 0, \infty, \infty, 1)$.

$$\begin{array}{lcl} \varphi_{\mathrm{PF}} & ::= & \langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{PF}} \mid \langle\varepsilon\rangle\bigwedge\{\psi_{\mathrm{PF}}, \psi_{\mathrm{PF}}, \ldots\} \mid \mathsf{T} \\ \psi_{\mathrm{PF}} & ::= & \neg\langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{T}} \mid \langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{T}} \mid \neg\langle\varepsilon\rangle\mathsf{T} \mid \langle\varepsilon\rangle\mathsf{T} \end{array}$$

Figure 5: Grammars induced by coordinates for linear-time notions of equivalence.

3.4. Linear-Time Notions

To define linear-time notions, the standard approach is to name special kinds of traces, paired or decorated with additional information to encode enabled or disabled actions. Instead of going through the motions of defining such objects, we argue that the modal characterizations induced by our coordinates match the right kinds of decorated traces.

For weak traces, Lemma 2.2 has established that the coordinate $e_{\mathrm{T}} = (\infty, 0, 0, 0, 0, 0, 0, 0)$ yields a characterization. For the other notions, we will remain less formal.

Example 3.4 (Failure equivalence and preorder)**.** The notion of (unstable) failure preorder (and its induced equivalence) is defined through $e_{\mathrm{F}} = (\infty, 0, 1, 0, 0, 0, 1, 1)$ and Definition 3.3 inducing $\mathcal{O}_{\mathrm{F}}$, the language given by the grammar:

$$\begin{array}{llcl} \mathcal{O}_{\mathrm{F}}: & \varphi_{\mathrm{F}} & ::= & \langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{F}} \mid \langle\varepsilon\rangle\bigwedge\{\psi_{\mathrm{F}}, \psi_{\mathrm{F}}, \ldots\} \mid \mathsf{T} \\ & \psi_{\mathrm{F}} & ::= & \neg\langle\varepsilon\rangle\langle a\rangle\mathsf{T} \mid \neg\langle\varepsilon\rangle\langle a\rangle\langle\varepsilon\rangle\mathsf{T} \mid \langle\varepsilon\rangle\mathsf{T} \mid \neg\langle\varepsilon\rangle\mathsf{T} \end{array}$$

The notion of stable failure preorder (and equivalence) is defined through $e_{\mathrm{F^s}} = (\infty, 0, 0, 1, 0, 0, 1, 1)$ and Definition 3.3 inducing $\mathcal{O}_{\mathrm{F^s}}$, the language given by the grammar:

$$\mathcal{O}_{\mathrm{F^s}}: \quad \varphi_{\mathrm{F^s}} \quad ::= \quad \langle\varepsilon\rangle\langle a\rangle\varphi_{\mathrm{F^s}} \mid \langle\varepsilon\rangle\bigwedge\{\neg\langle\tau\rangle, \psi_{\mathrm{F}}, \psi_{\mathrm{F}}, \ldots\} \mid \mathsf{T} \mid \langle\varepsilon\rangle\mathsf{T}$$

Intuitively, both grammars allow to observe a trace ending in one big conjunction of impossible actions. For (unstable) failures, this is a standard conjunction; for stable failures, the conjunction contains a negated τ-conjunct, enforcing stability. This mirrors exactly the standard definitions of weak unstable failures and stable failures.

Some of the productions do not add distinctive power: In particular, the last one of φ_{F} and the last three of ψ_{F} could be left out, and one would still obtain an equivalent modal characterization of failures.

They mostly appear out of consistence, as we assign T and $\langle\varepsilon\rangle\mathsf{T}$ expressiveness price $\mathbf{0}$. And this in turn is necessary to have the coordinates allow productions like the last one of $\varphi_{\mathrm{F^s}}$, where the possibility to end a trace observation lacking stabilization without this counting as a standard conjunction *does* make an important difference.

There is not much benefit in discussing the details for each linear-time notion. For completeness, we list the grammars for the notions of our weak spectrum in Figure 5. Also, we glimpse over ready similarity and 2-nested-similarity, which combine linear-time notions and similarity in unsurprising ways.

4. A Game of Distinguishing Capabilities

This section introduces a game to find out how two states can be distinguished in the silent-step spectrum: Attacker tries to implicitly construct a distinguishing formula, defender wants to prove that no such formula exists. The twist is that we use an *energy* game where energies ensure the possible formulas to lie in sublogics along the lines of the previous section.

4.1. Declining Energy Games

Equivalence problems of the strong linear-time–branching-time spectrum can be characterized as multi-dimensional declining energy games with special $\mathtt{min}$-operations between components as outlined in [9]. These games lie in the class of *Galois energy games,* which Lemke and Bisping [10] introduce as a generalization of multi-dimensional energy games. In this subsection, we revisit the definitions that we will need in this paper. For a more detailed presentation—in particular on how to compute attacker and defender winning budgets on this class of games—we refer to [10]. Formalized proofs can be found in [22].

Definition 4.1 (Energy updates)**.** The set of *energy updates,* $\mathbf{Up}$, contains vectors $(u_1, \dots, u_8) \in \mathbf{Up}$ where each component u_k is a symbol of the form

- $u_k \in \{-1, 0\}$ (relative update), or
- $u_k = \mathtt{min}_D$ where $D \subseteq \{1, \dots, 8\}$ and $k \in D$ (minimum selection update).

Applying an update to an energy, $\mathsf{upd}(e, u)$, where $e = (e_1, \dots, e_8) \in \mathbf{En}_\infty$ and $u = (u_1, \dots, u_8) \in \mathbf{Up}$, yields a new energy vector e' where kth components $e'_k := e_k + u_k$ for $u_k \in \mathbb{Z}$ and $e'_k := \min_{d \in D} e_d$ for $u_k = \mathtt{min}_D$. Updates that would cause any component to become negative are undefined, i.e., upd is a partial function.

Example 4.1. $\mathsf{upd}((2, 0, \infty, 0, 0, 0, 1, 1), (\mathtt{min}_{\{1,7\}}, 0, -1, 0, 0, 0, 0, -1))$ equals $(1, 0, \infty, 0, 0, 0, 1, 0)$.

Definition 4.2 (Games)**.** An 8-dimensional *declining energy game* $\mathcal{G} = (G, G_{\mathrm{d}}, \rightarrowtail, w)$ is played on a directed graph uniquely labeled by energy updates consisting of

- a set of *game positions* G, partitioned into
  - *defender positions* $G_{\mathrm{d}} \subseteq G$ and
  - *attacker positions* $G_{\mathrm{a}} := G \setminus G_{\mathrm{d}}$,
- a relation of *game moves* $\rightarrowtail \subseteq G \times G$, and
- a *weight function* for the moves $w : (\rightarrowtail) \to \mathbf{Up}$.

The notation $g \overset{u}{\rightarrowtail} g'$ stands for $g \rightarrowtail g'$ and $w(g, g') = u$.

In the games of [10], the attacker wins precisely if they can get the defender stuck without running out of energy. The energy budgets that suffice for the attacker to win from a game position can be characterized as follows:

Definition 4.3 (Winning budgets)**.** The attacker winning budgets $\mathsf{Win}_{\mathrm{a}}^{\mathcal{G}}$ per position of a game $\mathcal{G}$ are defined inductively by the rules:

$$\frac{g_{\mathrm{a}} \in G_{\mathrm{a}} \qquad g_{\mathrm{a}} \overset{u}{\rightarrowtail} g' \qquad \mathsf{upd}(e, u) \in \mathsf{Win}_{\mathrm{a}}^{\mathcal{G}}(g')}{e \in \mathsf{Win}_{\mathrm{a}}^{\mathcal{G}}(g_{\mathrm{a}})} \qquad \frac{g_{\mathrm{d}} \in G_{\mathrm{d}} \qquad \forall u, g'.\ g_{\mathrm{d}} \overset{u}{\rightarrowtail} g' \longrightarrow \mathsf{upd}(e, u) \in \mathsf{Win}_{\mathrm{a}}^{\mathcal{G}}(g')}{e \in \mathsf{Win}_{\mathrm{a}}^{\mathcal{G}}(g_{\mathrm{d}})}$$

The defender wins a budget from a position precisely if the attacker does not.

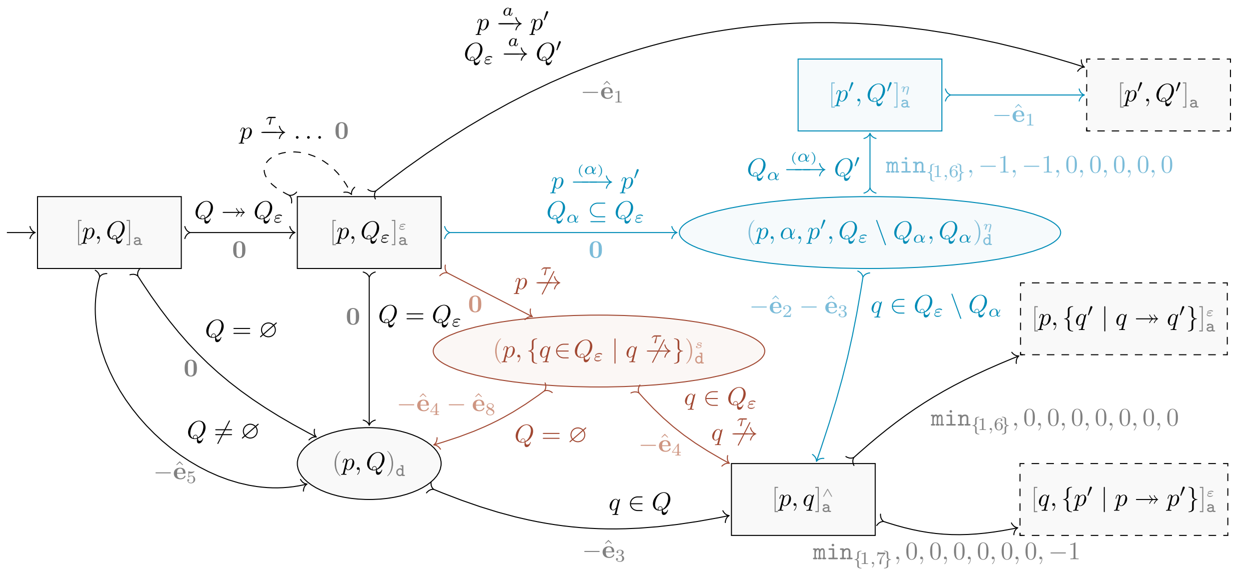

Figure 6: Schematic spectroscopy game $\mathcal{G}_\vartriangle$ of Definitions 4.4 (the black part), 4.5 (with position $(\cdots)^s_d$), and 4.6 (with positions $(\cdots)^s_d$, $(\cdots)^\eta_d$ and $[\cdots]^\eta_a$).

This is to say that the defender wins in the game if the attacker runs out of energy or moves, or if the game would take forever.

Remark 4.1. Often, winning budgets are rather defined through winning plays and strategies. Lemke [23] formally shows the equivalence of both approaches for games with an energy-bound attacker.

The definition of winning budgets implies that the attacker winning budgets are upward-closed, i.e., if $e \in \mathsf{Win}^{\mathcal{G}}_a(g)$ and $e' \geq e$, then $e' \in \mathsf{Win}^{\mathcal{G}}_a(g)$ [10, Lemma 9].

4.2. Delaying Observations in the Spectroscopy Energy Game

We begin with the part of the game that adds the concept of "delayed" attack positions to the "strong" spectroscopy game of [9]. It matches the black part of the $\mathsf{HML}_{\mathrm{srbb}}$-grammar of Definition 2.5. Figure 6 gives a schematic overview of the game rules, where the game would need to be further unfolded at the dashed nodes. Rectangular nodes belong to the attacker, elliptical ones to the defender. The attacker node with the dashed loop actually stands for a sequence of nodes. The colors differentiate the layers of following definitions and match the scheme of Definition 2.5 and Figure 4.

Definition 4.4 (Spectroscopy delay game)**.** For a system $\mathcal{S} = (\mathcal{P}, \Sigma, \rightarrow)$, the *spectroscopy delay energy game* $\mathcal{G}^{\mathcal{S}}_\varepsilon = (G, G_d, \rightarrowtail, w)$ consists of

- *attacker immediate positions* $[p, Q]_a \in G_a$,
- *attacker delayed positions* $[p, Q]^\varepsilon_a \in G_a$,
- *attacker conjunct positions* $[p, q]^\wedge_a \in G_a$,
- *defender conjunction positions* $(p, Q)_d \in G_d$,

where $p, q \in \mathcal{P}$, $Q \in \mathbf{2}^{\mathcal{P}}$, and nine kinds of moves:

- *delay* $[p, Q]_a \xrightarrowtail{0,0,0,0,0,0,0,0} [p, Q']^\varepsilon_a$ if $Q \twoheadrightarrow Q'$,
- *procrastination* $[p, Q]^\varepsilon_a \xrightarrowtail{0,0,0,0,0,0,0,0} [p', Q]^\varepsilon_a$ if $p \xrightarrow{\tau} p'$, $p \neq p'$,
- *observation* $[p, Q]^\varepsilon_a \xrightarrowtail{-1,0,0,0,0,0,0,0} [p', Q']_a$ if $p \xrightarrow{a} p'$, $Q \xrightarrow{a} Q'$, $a \neq \tau$,
- *finishing* $[p, \varnothing]_a \xrightarrowtail{0,0,0,0,0,0,0,0} (p, \varnothing)_d$,

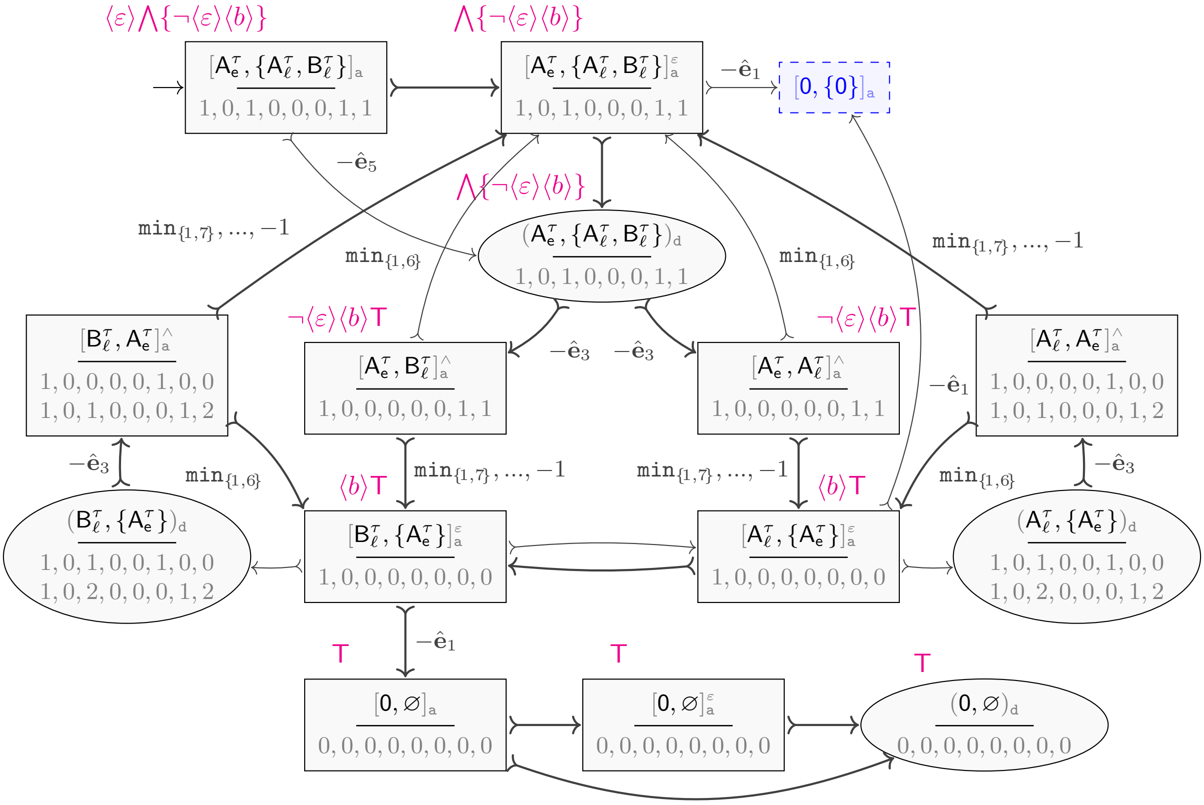

Figure 7: Spectroscopy delay game $\mathcal{G}_\varepsilon$ from $[\mathsf{A}^\tau_\mathsf{e}, \{\mathsf{A}^\tau_\ell, \mathsf{B}^\tau_\ell\}]_\mathsf{a}$ for Example 4.2. Each position names minimal attacker-winning budgets (due to the thick arrows) and corresponding distinguishing formulas (pink). Zeroes and **0**-updates are omitted for readability. Also, the game graph under defender-won reflexive position $[0, \{0\}]_\mathsf{a}$ (dashed in blue) is omitted.

- *immediate conj.* $[p, Q]_\mathsf{a} \xrightarrow{0,0,0,0,-1,0,0,0} (p, Q)_\mathsf{d}$ if $Q \neq \varnothing$,
- *late conj.* $[p, Q]^\varepsilon_\mathsf{a} \xrightarrow{0,0,0,0,0,0,0,0} (p, Q)_\mathsf{d}$,
- *conj. answer* $(p, Q)_\mathsf{d} \xrightarrow{0,0,-1,0,0,0,0,0} [p, q]^\wedge_\mathsf{a}$ if $q \in Q$,
- *positive conjunct* $[p, q]^\wedge_\mathsf{a} \xrightarrow{\mathtt{min}_{\{1,6\}},0,0,0,0,0,0,0} [p, Q]^\varepsilon_\mathsf{a}$ if $\{q\} \twoheadrightarrow Q$,
- *negative conjunct* $[p, q]^\wedge_\mathsf{a} \xrightarrow{\mathtt{min}_{\{1,7\}},0,0,0,0,0,0,-1} [q, Q]^\varepsilon_\mathsf{a}$ if $\{p\} \twoheadrightarrow Q$ and $p \neq q$.

Example 4.2. Starting at $\mathsf{P}^\tau_\mathsf{e}$ and P^τ_ℓ of Example 2.1 with energy $(2,0,1,0,0,0,1,1)$, the attacker can move with $[\mathsf{P}^\tau_\mathsf{e}, \{\mathsf{P}^\tau_\ell\}]_\mathsf{a} \xrightarrow{\text{delay}}\xrightarrow{\text{observation}} [\mathsf{A}^\tau_\mathsf{e}, \{\mathsf{A}^\tau_\ell, \mathsf{B}^\tau_\ell\}]_\mathsf{a}$. (For readability, we label the moves by the names of their rules.) This uses up $\hat{\mathbf{e}}_1$ energy leading to level $(1,0,1,0,0,0,1,1)$.

Figure 7 shows how the attacker can win from there. The attacker chooses a delay move and yields to the defender $(\mathsf{A}^\tau_\mathsf{e}, \{\mathsf{A}^\tau_\ell, \mathsf{B}^\tau_\ell\})_\mathsf{d}$. If the defender selects B^τ_ℓ, bringing the energy to $(1,0,0,0,0,0,1,1)$, the attacker wins by $[\mathsf{A}^\tau_\mathsf{e}, \mathsf{B}^\tau_\ell]^\wedge_\mathsf{a} \xrightarrow{\text{negative conjunct}} [\mathsf{B}^\tau_\ell, \mathsf{A}^\tau_\mathsf{e}]^\varepsilon_\mathsf{a} \xrightarrow{\text{observation}}\xrightarrow{\text{finishing}} (0, \varnothing)_\mathsf{d} \not\rightarrowtail$. For the defender choosing A^τ_ℓ, a similar attack works due to $[\mathsf{A}^\tau_\ell, \mathsf{A}^\tau_\mathsf{e}]^\varepsilon_\mathsf{a} \xrightarrow{\text{procrastination}} [\mathsf{B}^\tau_\ell, \mathsf{A}^\tau_\mathsf{e}]^\varepsilon_\mathsf{a}$. Thus, the attacker wins the game.

The tree of winning moves corresponds to formula $\varphi_\tau = \langle\varepsilon\rangle\langle op\rangle\langle\varepsilon\rangle\bigwedge\{\neg\langle\varepsilon\rangle\langle b\rangle\mathsf{T}\}$ and budget of Example 3.1. This is no coincidence, but rather our core design principle for game moves. As we will prove in Section 5, attacker's winning moves match distinguishing $\mathsf{HML}_{\text{srbb}}$-formulas and their prices.

Note that the attacker would not win if any component of the starting energy vector were lower. For example, $e_\mathrm{T} = (\infty,0,0,0,0,0,0,0) \notin \mathsf{Win}_\mathsf{a}([\mathsf{P}^\tau_\mathsf{e}, \{\mathsf{P}^\tau_\ell\}]_\mathsf{a})$ corresponds to weak trace preorder, $\mathsf{P}^\tau_\mathsf{e} \preceq_\mathrm{T} \mathsf{P}^\tau_\ell$.

Remark 4.2. The construction principle of the spectroscopy game in Definition 4.4 is to follow the $\mathsf{HML}_{\text{srbb}}$-grammar of Definition 2.5. For Theorem 5.1, we will prove how attacker positions correspond to nonterminals and winning budgets to formula prices. Preliminarily, let us work with the intuition:

- Attacker immediate positions $[\ldots]_{\mathtt{a}}$ correspond to φ-nonterminals. When internal behavior $\twoheadrightarrow$ justifies an attacker move, there appears an $\langle\varepsilon\rangle$-modality in the formula.
- Attacker delayed positions $[\ldots]^{\varepsilon}_{\mathtt{a}}$ represent χ-terms. A visible step $\xrightarrow{a}$ in the observation move matches a $\langle a\rangle$-modality in the formula.
- Attacker conjunct positions $[\ldots]^{\wedge}_{\mathtt{a}}$ express ψ-conjuncts. When p and q swap sides, the move represents a negation $\neg$.
- Defender conjunction positions $(\ldots)_{\mathtt{d}}$ correspond to conjunctions in formulas $\bigwedge\{\ldots\}$.

4.3. Covering Stable Failures and Conjunctions

In order to cover "stable" and "stability-respecting" equivalences, we must separately count stable conjunctions.

Definition 4.5 (Spectroscopy stability game)**.** The *stability game* $\mathcal{G}^{\mathcal{S}}_{s}$ extends the delay game $\mathcal{G}^{\mathcal{S}}_{\varepsilon}$ of Definition 4.4 by

- *defender stable conjunction positions* $\quad (p, Q)^{s}_{\mathtt{d}} \in G_{\mathrm{d}}$,

where $p \in \mathcal{P}$, $Q \in \mathbf{2}^{\mathcal{P}}$, and three kinds of moves:

- *stable conj.* $\quad [p, Q]^{\varepsilon}_{\mathtt{a}} \xrightarrow{0,0,0,0,0,0,0,0} (p, Q')^{s}_{\mathtt{d}}$ if $Q' = \{q \in Q \mid q \stackrel{\tau}{\not\rightarrow}\}$, $p \stackrel{\tau}{\not\rightarrow}$,
- *conj. stable answer* $\quad (p, Q)^{s}_{\mathtt{d}} \xrightarrow{0,0,0,-1,0,0,0,0} [p, q]^{\wedge}_{\mathtt{a}}$ if $q \in Q$,
- *stable finishing* $\quad (p, Q)^{s}_{\mathtt{d}} \xrightarrow{0,0,0,-1,0,0,0,-1} (p, \varnothing)_{\mathtt{d}}$.

In principle, we add a move to enter a defender stable conjunction position and a move to leave it, similar to the defender conjunction positions in Definition 4.4.

Example 4.3. Note that these new rules allow no new (incomparable) wins for the attacker in Example 4.2. Therefore, *stable bisimulation* is another finest preorder (and equivalence) for the example processes because $e_{\mathrm{SB}} \notin \mathsf{Win}_{\mathrm{a}}([\mathsf{P}^{\tau}_{\mathsf{e}}, \{\mathsf{P}^{\tau}_{\ell}\}]_{\mathtt{a}})$ for $\mathcal{G}_s$.

4.4. Extending to Branching Bisimulation

One last kind of distinctions is necessary to characterize *branching bisimilarity,* the strongest common abstraction of bisimilarity for systems with silent steps: its characteristic branching conjunctions.

Definition 4.6 (Weak spectroscopy game)**.** The *weak spectroscopy energy game* $\mathcal{G}^{\mathcal{S}}_{\vartriangle}$ extends Definition 4.5 by

- *defender branching positions* $\quad (p, \alpha, p', Q, Q_\alpha)^{\eta}_{\mathtt{d}} \in G_{\mathrm{d}}$,
- *attacker branching positions* $\quad [p, Q]^{\eta}_{\mathtt{a}} \in G_{\mathrm{a}}$,

where $p, p' \in \mathcal{P}$ and $Q, Q_\alpha \in \mathbf{2}^{\mathcal{P}}$ as well as $\alpha \in \Sigma$, and four kinds of moves:

- *branching conj.* $\quad [p, Q]^{\varepsilon}_{\mathtt{a}} \xrightarrow{0,0,0,0,0,0,0,0} (p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^{\eta}_{\mathtt{d}}$ if $p \xrightarrow{(\alpha)} p'$, $Q_\alpha \subseteq Q$,
- *branch. answer* $\quad (p, \alpha, p', Q, Q_\alpha)^{\eta}_{\mathtt{d}} \xrightarrow{0,-1,-1,0,0,0,0,0} [p, q]^{\wedge}_{\mathtt{a}}$ if $q \in Q$,
- *branch. observation* $\quad (p, \alpha, p', Q, Q_\alpha)^{\eta}_{\mathtt{d}} \xrightarrow{\mathtt{min}_{\{1,6\}},-1,-1,0,0,0,0,0} [p', Q']^{\eta}_{\mathtt{a}}$ with $Q_\alpha \xrightarrow{(\alpha)} Q'$,
- *branch. accounting* $\quad [p, Q]^{\eta}_{\mathtt{a}} \xrightarrow{-1,0,0,0,0,0,0,0} [p, Q]_{\mathtt{a}}$.

Intuitively, in a branching conjunction move, the attacker picks a step $p \xrightarrow{\alpha} p'$ and claims for some states $Q_\alpha \subseteq Q$ that they cannot immediately simulate this α-step. For the remaining states in $Q \setminus Q_\alpha$, the attacker claims that p can be distinguished from them by other (possibly negative) delayed observations. The defender then chooses which claim to counter.

Example 4.4. Consider the CCS processes $a + \tau.b + b$ and $a + \tau.b$. The first process explicitly allows action b to happen before deciding against a. To weak bisimilarity, for instance, this is transparent. To more branching-aware notions, it constitutes a difference.

The two processes can be distinguished as follows in the weak spectroscopy game with energy budget $(1, 1, 1, 0, 0, 1, 0, 0)$: First, the attacker enters a defender branching position $[a + \tau.b + b, \{a + \tau.b\}]_a \xrightarrow{\text{delay}} [a + \tau.b + b, \{a + \tau.b, b\}]^{\varepsilon}_a \xrightarrow{\text{branching conjunction}} (a + \tau.b + b, b, 0, \{b\}, \{a + \tau.b\})^{\eta}_d$. The defender can then pick between two losing options:

- $(\cdots)^{\eta}_d \xrightarrow{\text{branching answer}} [a+\tau.b+b, b]^{\wedge}_a$: Attacker responds $[\cdots]^{\wedge}_a \xrightarrow{\text{positive conjunct}} \xrightarrow{a\text{-observation}} \xrightarrow{\text{finishing}} (0, \varnothing)_d$, which corresponds to formula $\langle\varepsilon\rangle\langle a\rangle\mathsf{T}$.
- $(\cdots)^{\eta}_d \xrightarrow{\text{branching observation}} [0, \varnothing]^{\eta}_a$: Attacker replies $[\cdots]^{\eta}_a \xrightarrow{\text{branching accounting}} \xrightarrow{\text{finishing}} (0, \varnothing)_d$, which corresponds to the $(b)\mathsf{T}$-observation in the context of a branching conjunction.

Taken together, the attacker wins this game constellation with a strategy that corresponds to the formula $\langle\varepsilon\rangle\bigwedge\{(b)\mathsf{T}, \langle\varepsilon\rangle\langle a\rangle\mathsf{T}\}$.

The formula disproves η-simulation preorder and thus branching bisimilarity. However, the two processes are (stability-respecting) delay-bisimilar as there are no delay bisimulation formulas to distinguish them.

5. Correctness

We now state in what sense winning energy levels and equivalences coincide in the context of a transition system $\mathcal{S} = (\mathcal{P}, \Sigma, \rightarrow)$.

Theorem 5.1 (Correctness). *For all $e \in \mathbf{En}_\infty$, $p \in \mathcal{P}$, $Q \in \mathbf{2}^{\mathcal{P}}$, the following are equivalent:*[6]

1. *There exists a formula $\varphi \in \mathsf{HML}_{\mathrm{srbb}}$ with price $\mathsf{expr}(\varphi) \leq e$ that distinguishes p from Q.*
2. *Attacker wins $\mathcal{G}^{\mathcal{S}}_{\vartriangle}$ from $[p, Q]_a$ with budget e (that is, $e \in \mathsf{Win}_a^{\mathcal{G}^{\mathcal{S}}_{\vartriangle}}([p, Q]_a)$).*

With Definition 3.4, this means that, for a notion of equivalence N with coordinate e_N in Figure 4, $p \preceq_N q$ precisely if the defender wins, that is, if $e_N \notin \mathsf{Win}_a([p, \{q\}]_a)$.

The proof of the theorem is given through the following three lemmas. The direction from (1) to (2) is covered by Lemma 5.1 when combined with the upward-closedness of attacker winning budgets. From (2) to (1), the link is established through *strategy formulas* by Lemmas 5.2 and 5.3.

5.1. Distinguishing Formulas Imply Attacker-Winning Budgets

Lemma 5.1. *If $\varphi \in \mathsf{HML}_{\mathrm{srbb}}$ distinguishes p from Q, then $\mathsf{expr}(\varphi) \in \mathsf{Win}_a([p, Q]_a)$.*[7]

Proof. If $Q = \varnothing$, the lemma is very easy to prove. So let us assume that $Q \neq \varnothing$ for the rest. To get an inductive property, we actually prove the following property:

1. If $\varphi \in \mathsf{HML}_{\mathrm{srbb}}$ distinguishes p from $Q \neq \varnothing$, then $\mathsf{expr}(\varphi) \in \mathsf{Win}_a([p, Q]_a)$;
2. If χ distinguishes p from $Q \neq \varnothing$ and Q is closed under $\twoheadrightarrow$ (that is $Q \twoheadrightarrow Q$), then $\mathsf{expr}^{\varepsilon}(\chi) \in \mathsf{Win}_a([p, Q]^{\varepsilon}_a)$;
3. If ψ distinguishes p from q, then $\mathsf{expr}^{\wedge}(\psi) \in \mathsf{Win}_a([p, q]^{\wedge}_a)$.
4. If the standard conjunction $\bigwedge\Psi$ distinguishes p from $Q \neq \varnothing$, then $\mathsf{expr}^{\varepsilon}(\bigwedge\Psi) \in \mathsf{Win}_a((p, Q)_d)$;
5. If $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi$ distinguishes p from $Q \neq \varnothing$ and the processes in Q are stable, then $\mathsf{expr}^{\varepsilon}(\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi) \in \mathsf{Win}_a((p, Q)^{s}_d)$;

[6] theorem `Silent_Step_Spectroscopy.lts_tau.spectroscopy_game_correctness`
[7] lemma `Distinction_Implies_Winning_Budgets.lts_tau.distinction_implies_winning_budgets`

6. If $\bigwedge\{(\alpha)\varphi'\} \cup \Psi$ distinguishes p from Q, then, for any $p \xrightarrow{(\alpha)} p' \in [\![\varphi']\!]$ and $Q_\alpha = Q \setminus [\![(\alpha)\varphi']\!]$, $\mathsf{expr}^\varepsilon(\bigwedge\{(\alpha)\varphi'\} \cup \Psi) \in \mathsf{Win}_\mathrm{a}((p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^\eta_\mathrm{d})$.

We prove this by mutual induction over the structure of φ, χ, and ψ.

1. Assume φ distinguishes p from $Q \neq \varnothing$.

   $\varphi = \langle\varepsilon\rangle\chi$**:** That means that there exists $p \twoheadrightarrow p' \in [\![\chi]\!]$ and $Q' \cap [\![\chi]\!] = \varnothing$ for $Q \twoheadrightarrow Q'$. Therefore, χ distinguishes p' from Q' and $Q' \twoheadrightarrow Q'$. By induction hypothesis we conclude that $\mathsf{expr}^\varepsilon(\chi) \in \mathsf{Win}_\mathrm{a}([p', Q']^\varepsilon_\mathrm{a})$.

   There are moves $[p, Q]_\mathrm{a} \xrightarrow{\text{delay}} [p, Q']^\varepsilon_\mathrm{a} \overset{\text{procrastination}}{\rightarrowtail \cdots \rightarrowtail} [p', Q']^\varepsilon_\mathrm{a}$. Using Definition 4.3 over these moves, we can conclude that $\mathsf{expr}^\varepsilon(\chi) \in \mathsf{Win}_\mathrm{a}([p, Q]_\mathrm{a})$. We get the result because $\mathsf{expr}(\varphi) = \mathsf{expr}^\varepsilon(\chi)$.

   $\varphi = \bigwedge\Psi$**:** There is the move $[p, Q]_\mathrm{a} \xrightarrow{\text{immediate conj.}} (p, Q)_\mathrm{d}$. By induction hypothesis we conclude that $\mathsf{expr}^\varepsilon(\bigwedge\Psi) \in \mathsf{Win}_\mathrm{a}((p, Q)_\mathrm{d})$. Using Definition 4.3 we immediately get that $\mathsf{expr}(\varphi) = \mathsf{expr}^\varepsilon(\varphi) + \hat{\mathbf{e}}_5 \in \mathsf{Win}_\mathrm{a}([p, Q]_\mathrm{a})$.

2. Assume χ distinguishes p from $Q \neq \varnothing$ (and $Q \twoheadrightarrow Q$).

   $\chi = \langle a\rangle\varphi'$**:** That means that there exists $p' \in [\![\varphi']\!]$ such that $p \xrightarrow{a} p'$. On the other hand, $Q' \cap [\![\varphi']\!] = \varnothing$, where $Q \xrightarrow{a} Q'$, and therefore φ' distinguishes p' from Q'.

   Now there is the move $[p, Q]^\varepsilon_\mathrm{a} \xrightarrow{\text{observation}} [p', Q']_\mathrm{a}$. By induction hypothesis we conclude that $\mathsf{expr}(\varphi') \in \mathsf{Win}_\mathrm{a}([p', Q']_\mathrm{a})$. Because we can calculate $\mathsf{expr}^\varepsilon(\langle a\rangle\varphi') := \hat{\mathbf{e}}_1 + \mathsf{expr}(\varphi')$, we know $\mathsf{upd}(\mathsf{expr}^\varepsilon(\chi), -\hat{\mathbf{e}}_1) = \mathsf{expr}(\varphi')$. With Definition 4.3, we get $\mathsf{expr}^\varepsilon(\chi) \in \mathsf{Win}_\mathrm{a}([p, Q]^\varepsilon_\mathrm{a})$.

   $\chi = \bigwedge\Psi$**:** There is the move $[p, Q]^\varepsilon_\mathrm{a} \xrightarrow{\text{late conj.}} (p, Q)_\mathrm{d}$; we use the proof for $(p, Q)_\mathrm{d}$ that follows in (case 4) and Definition 4.3 to then get $\mathsf{expr}^\varepsilon(\chi) \in \mathsf{Win}_\mathrm{a}([p, Q]^\varepsilon_\mathrm{a})$.

   $\chi = \bigwedge\{\neg\langle\tau\rangle\top\} \cup \Psi$**:** There is the move $[p, Q]^\varepsilon_\mathrm{a} \xrightarrow{\text{stable conj.}} (p, Q')^s_\mathrm{d}$, where $Q' = \{q \in Q \mid q \not\xrightarrow{\tau}\}$. If Q' is not empty, we argue as in the previous case using (case 5).

   If Q' is empty, there is only the move $(p, Q')^s_\mathrm{d} = (p, \varnothing)^s_\mathrm{d} \xrightarrow{\text{stable finishing}} (p, \varnothing)_\mathrm{d}$. The latter position is stuck, so $\mathsf{Win}_\mathrm{a}((p, \varnothing)_\mathrm{d}) = \mathbf{En}_\infty$ and by Definition 4.3, $e \in \mathsf{Win}_\mathrm{a}((p, \varnothing)^s_\mathrm{d})$ for all $e \geq \hat{\mathbf{e}}_4 + \hat{\mathbf{e}}_8$. Because $\mathsf{expr}^\varepsilon(\chi) \geq \mathsf{expr}^\varepsilon(\bigwedge\{\neg\langle\tau\rangle\top\}) = \hat{\mathbf{e}}_4 + \hat{\mathbf{e}}_8$, we get the result.

   $\chi = \bigwedge\{(\alpha)\varphi'\} \cup \Psi$**:** Note that there must exist $p \xrightarrow{(\alpha)} p' \in [\![\varphi']\!]$ (otherwise $p \notin [\![(\alpha)\varphi']\!] \supseteq [\![\chi]\!]$, so χ would not distinguish p from anything). Pick such a p', and set $Q_\alpha = Q \setminus [\![(\alpha)\varphi']\!]$. Then there is the move $[p, Q]^\varepsilon_\mathrm{a} \xrightarrow{\text{branching conj.}} (p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^\eta_\mathrm{d}$; so we can use the proof for $(p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^\eta_\mathrm{d}$ that follows in (case 6) and Definition 4.3 to get $\mathsf{expr}^\varepsilon(\chi) \in \mathsf{Win}_\mathrm{a}([p, Q]^\varepsilon_\mathrm{a})$.

3. Assume ψ distinguishes p from q.

   $\psi = \langle\varepsilon\rangle\chi$**:** That means that there exists $p \twoheadrightarrow p' \in [\![\chi]\!]$ and $Q' \cap [\![\chi]\!] = \varnothing$ for $\{q\} \twoheadrightarrow Q'$. Therefore, χ distinguishes p' from Q' and $Q' \twoheadrightarrow Q'$. By induction hypothesis we conclude that $\mathsf{expr}^\varepsilon(\chi) \in \mathsf{Win}_\mathrm{a}([p', Q']^\varepsilon_\mathrm{a})$.

   Now there is a move sequence $[p, q]^\wedge_\mathrm{a} \xrightarrow{\text{positive conjunct}} [p, Q']^\varepsilon_\mathrm{a} \overset{\text{procrastination}}{\rightarrowtail \cdots \rightarrowtail} [p', Q']^\varepsilon_\mathrm{a}$. Using Definition 4.3 over the procrastination moves, we can conclude $\mathsf{expr}(\langle\varepsilon\rangle\chi) = \mathsf{expr}^\varepsilon(\chi) \in \mathsf{Win}_\mathrm{a}([p, Q']^\varepsilon_\mathrm{a})$. Calculation shows $\mathsf{upd}(\mathsf{expr}^\wedge(\psi), (\mathtt{min}_{\{1,6\}}, 0, 0, 0, 0, 0, 0, 0)) \geq \mathsf{expr}^\varepsilon(\chi)$, and this allows to apply Definition 4.3 and get the result.

   $\psi = \neg\langle\varepsilon\rangle\chi$**:** That means that there exists $q \twoheadrightarrow q' \in [\![\chi]\!]$ and $P' \cap [\![\chi]\!] = \varnothing$ for $\{p\} \twoheadrightarrow P'$. Therefore, χ distinguishes q' from P' and $P' \twoheadrightarrow P'$. By induction hypothesis we conclude that $\mathsf{expr}^\varepsilon(\chi) \in \mathsf{Win}_\mathrm{a}([q', P']^\varepsilon_\mathrm{a})$. Again, we apply Definition 4.3 to obtain the result, using a similar calculation as in the previous case to show $\mathsf{upd}(\mathsf{expr}^\wedge(\psi), (\mathtt{min}_{\{1,7\}}, 0, 0, 0, 0, 0, 0, -1)) \geq \mathsf{expr}^\varepsilon(\chi)$.

4. Assume the standard conjunction $\bigwedge\Psi$ distinguishes p from Q.

We can find, for every $q \in Q$, some $\psi_q \in \Psi$ such that $q \notin [\![\psi_q]\!]$ (so $\Psi \neq \varnothing$). Choose one such covering of ψ_qs. Let $\Psi' := \{\psi_q \mid q \in Q\} \subseteq \Psi$. Each ψ_q either has the form $\langle\varepsilon\rangle\chi_q$ or $\neg\langle\varepsilon\rangle\chi_q$. It must be the case that $p \in [\![\bigwedge_{q\in Q}\psi_q]\!]$ and $Q \cap [\![\bigwedge_{q\in Q}\psi_q]\!] = \varnothing$.

Now there are the moves $(p,Q)_{\mathrm{d}} \xrightarrow{\text{conj. answer}} [p,q]^{\wedge}_{\mathrm{a}}$ for all $q \in Q$. We have to show that $e_0 := \mathsf{expr}^{\varepsilon}(\bigwedge\Psi) = \hat{\mathbf{e}}_3 + \sup\{\mathsf{expr}^{\wedge}(\psi) \mid \psi \in \Psi\} \in \mathsf{Win}_{\mathrm{a}}((p,Q)_{\mathrm{d}})$. As $\mathsf{Win}_{\mathrm{a}}((p,Q)_{\mathrm{d}})$ is upwards-closed, we can restrict the supremum to Ψ' instead of Ψ, so it suffices to prove that $\sup\{\mathsf{expr}^{\wedge}(\psi) + \hat{\mathbf{e}}_3 \mid \psi \in \Psi'\} \in \mathsf{Win}_{\mathrm{a}}((p,Q)_{\mathrm{d}})$. Now, to show this using Definition 4.3, we have to quantify over all game moves from $(p,Q)_{\mathrm{d}}$, i.e. over all conjunction answers, which lead to the positions $[p,q]^{\wedge}_{\mathrm{a}}$ for $q \in Q$. We have $\mathsf{upd}(e_0, -\hat{\mathbf{e}}_3) \geq \mathsf{expr}^{\wedge}(\psi_q)$, and by induction hypothesis know $\mathsf{expr}^{\wedge}(\psi_q) \in \mathsf{Win}_{\mathrm{a}}([p,q]^{\wedge}_{\mathrm{a}})$. Applying Definition 4.3 immediately leads to the desired result.

5. Assume $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi$ distinguishes p from $Q \neq \varnothing$, where p and the processes in Q are stable.

   The defender always has the move $(p,Q)^{s}_{\mathrm{d}} \xrightarrow{\text{stable finishing}} (p,\varnothing)_{\mathrm{d}}$, which ensures that the attacker needs at least $\hat{\mathbf{e}}_4 + \hat{\mathbf{e}}_8$ energy to win. But this does not endanger the proof goal, because the formula form ensures $\mathsf{expr}^{\varepsilon}(\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi) \geq \hat{\mathbf{e}}_4 + \hat{\mathbf{e}}_8$.

   We choose ψ_q (for every $q \in Q$) and Ψ' as in the previous case.

   Now there are the moves $(p,Q)^{s}_{\mathrm{d}} \xrightarrow{\text{conj. s-answer}} [p,q]^{\wedge}_{\mathrm{a}}$ for all $q \in Q$. As in the previous case, we can show that $\sup\{\mathsf{expr}^{\wedge}(\psi) + \hat{\mathbf{e}}_4 \mid \psi \in \Psi'\}$ suffices for the attacker to win any of these moves.

   The attacker budget $\mathsf{expr}^{\varepsilon}(\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi) = \hat{\mathbf{e}}_4 + \sup(\{\hat{\mathbf{e}}_8\} \cup \{\mathsf{expr}^{\wedge}(\psi) \mid \psi \in \Psi\})$ dominates both cases of defender moves, and thus $\mathsf{expr}^{\varepsilon}(\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi) \in \mathsf{Win}_{\mathrm{a}}((p,Q)^{s}_{\mathrm{d}})$.

6. Assume $\bigwedge\{(\alpha)\varphi'\} \cup \Psi$ distinguishes p from Q, $p \xrightarrow{(\alpha)} p' \in [\![\varphi']\!]$ and $Q_\alpha = Q \setminus [\![\langle\alpha\rangle\varphi']\!]$.

   There are the moves $(p,\alpha,p',Q\setminus Q_\alpha,Q_\alpha)^{\eta}_{\mathrm{d}} \xrightarrow{\text{br. answer}} [p,q]^{\wedge}_{\mathrm{a}}$ for all $q \in Q \setminus Q_\alpha$. Additionally, there are the moves $(p,\alpha,p',Q\setminus Q_\alpha,Q_\alpha)^{\eta}_{\mathrm{d}} \xrightarrow{\text{br. observation}} [p',Q']^{\eta}_{\mathrm{a}} \xrightarrow{\text{br. accounting}} [p',Q']_{\mathrm{a}}$ for $Q_\alpha \xrightarrow{(\alpha)} Q'$, and φ' distinguishes p' from Q'.

   We must show that $e_0 := \mathsf{expr}^{\varepsilon}(\bigwedge\{(\alpha)\varphi'\}\cup\Psi) = \hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3+\sup(\{\mathsf{expr}^{\wedge}(\langle\varepsilon\rangle(\alpha)\varphi')\}\cup\{\mathsf{expr}^{\wedge}(\psi) \mid \psi \in \Psi\}) \in \mathsf{Win}_{\mathrm{a}}((p,\alpha,p',Q\setminus Q_\alpha,Q_\alpha)^{\eta}_{\mathrm{d}})$. For the branching answer moves, this proceeds exactly as in the previous cases. For the branching observation move, we have to show that $e_2 := \mathsf{upd}(\mathsf{upd}(e_0, (\mathtt{min}_{\{1,6\}}, -1, -1, 0, 0, 0, 0, 0)), -\hat{\mathbf{e}}_1) \in \mathsf{Win}_{\mathrm{a}}([p',Q']_{\mathrm{a}})$. We know $e_2 \geq \mathsf{expr}(\varphi')$. Moreover, we get $\mathsf{expr}(\varphi') \in \mathsf{Win}_{\mathrm{a}}([p',Q']_{\mathrm{a}})$ by induction hypothesis (or, trivially, if $Q' = \varnothing$). This suffices to apply Definition 4.3 and get the result. □

5.2. Winning Attacks Imply Cheap Distinguishing Formulas

For this direction of the proof, we make explicit what formulas attacker's strategies correspond to.

Definition 5.1 (Strategy formulas)**.** The set of *attacker strategy formulas* Strat for a $\mathcal{G}_\triangle$-position with given energy level e is derived from the sets of winning budgets, $\mathsf{Win}_{\mathrm{a}}$, inductively according to the rules in Figure 8.

As an example how to read the rules of Figure 8, *procr* states that if there is a move $[p,Q]^{\varepsilon}_{\mathrm{a}} \xrightarrow{u} [p',Q]^{\varepsilon}_{\mathrm{a}}$ (based on Definition 4.4, this must be a procrastination move), and the strategy formulas of the latter position contain χ, then also the strategy formulas of the former position contain χ.

Lemma 5.2. *If* $e \in \mathsf{Win}_{\mathrm{a}}([p,Q]_{\mathrm{a}})$*, then there is* $\varphi \in \mathsf{Strat}([p,Q]_{\mathrm{a}}, e)$ *with* $\mathsf{expr}(\varphi) \leq e$.[8]

Proof. We prove a more detailed result, namely:

1. If $e \in \mathsf{Win}_{\mathrm{a}}([p,Q]_{\mathrm{a}})$, then there is $\varphi \in \mathsf{Strat}([p,Q]_{\mathrm{a}}, e)$ with price $\mathsf{expr}(\varphi) \leq e$;

[8] lemma `Strategy_Formulas.lts_tau.winning_budget_implies_strategy_formula`

$$\text{delay}\ \dfrac{[p,Q]_{\mathrm{a}} \overset{u}{\rightarrowtail} [p,Q']^{\varepsilon}_{\mathrm{a}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}([p,Q']^{\varepsilon}_{\mathrm{a}}) \qquad \chi \in \mathsf{Strat}([p,Q']^{\varepsilon}_{\mathrm{a}}, e')}{\langle\varepsilon\rangle\chi \in \mathsf{Strat}([p,Q]_{\mathrm{a}}, e)}$$

$$\text{procr}\ \dfrac{[p,Q]^{\varepsilon}_{\mathrm{a}} \overset{u}{\rightarrowtail} [p',Q]^{\varepsilon}_{\mathrm{a}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}([p',Q]^{\varepsilon}_{\mathrm{a}}) \qquad \chi \in \mathsf{Strat}([p',Q]^{\varepsilon}_{\mathrm{a}}, e')}{\chi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)}$$

$$\text{observation}\ \dfrac{e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}([p',Q']_{\mathrm{a}}) \qquad \begin{matrix}[p,Q]^{\varepsilon}_{\mathrm{a}} \overset{u}{\rightarrowtail} [p',Q']_{\mathrm{a}} \\ p \xrightarrow{a} p' \qquad Q \xrightarrow{a} Q'\end{matrix} \qquad \varphi \in \mathsf{Strat}([p',Q']_{\mathrm{a}}, e')}{\langle a\rangle\varphi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)}$$

$$\text{immediate conj}\ \dfrac{[p,Q]_{\mathrm{a}} \overset{u}{\rightarrowtail} (p,Q)_{\mathrm{d}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}((p,Q)_{\mathrm{d}}) \qquad \varphi \in \mathsf{Strat}((p,Q)_{\mathrm{d}}, e')}{\varphi \in \mathsf{Strat}([p,Q]_{\mathrm{a}}, e)}$$

$$\text{late conj}\ \dfrac{[p,Q]^{\varepsilon}_{\mathrm{a}} \overset{u}{\rightarrowtail} (p,Q)_{\mathrm{d}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}((p,Q)_{\mathrm{d}}) \qquad \chi \in \mathsf{Strat}((p,Q)_{\mathrm{d}}, e')}{\chi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)}$$

$$\text{conj}\ \dfrac{\forall q \in Q.\ (p,Q)_{\mathrm{d}} \overset{u_q}{\rightarrowtail} [p,q]^{\wedge}_{\mathrm{a}} \ \wedge\ e_q = \mathsf{upd}(e,u_q) \in \mathsf{Win}_{\mathrm{a}}([p,q]^{\wedge}_{\mathrm{a}}) \ \wedge\ \psi_q \in \mathsf{Strat}([p,q]^{\wedge}_{\mathrm{a}}, e_q)}{\bigwedge\{\psi_q \mid q \in Q\} \in \mathsf{Strat}((p,Q)_{\mathrm{d}}, e)}$$

$$\text{pos}\ \dfrac{[p,q]^{\wedge}_{\mathrm{a}} \overset{u}{\rightarrowtail} [p,Q']^{\varepsilon}_{\mathrm{a}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}([p,Q']^{\varepsilon}_{\mathrm{a}}) \qquad \chi \in \mathsf{Strat}([p,Q']^{\varepsilon}_{\mathrm{a}}, e')}{\langle\varepsilon\rangle\chi \in \mathsf{Strat}([p,q]^{\wedge}_{\mathrm{a}}, e)}$$

$$\text{neg}\ \dfrac{[p,q]^{\wedge}_{\mathrm{a}} \overset{u}{\rightarrowtail} [q,P']^{\varepsilon}_{\mathrm{a}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}([q,P']^{\varepsilon}_{\mathrm{a}}) \qquad \chi \in \mathsf{Strat}([q,P']^{\varepsilon}_{\mathrm{a}}, e')}{\neg\langle\varepsilon\rangle\chi \in \mathsf{Strat}([p,q]^{\wedge}_{\mathrm{a}}, e)}$$

$$\text{stable}\ \dfrac{[p,Q]^{\varepsilon}_{\mathrm{a}} \overset{u}{\rightarrowtail} (p,Q')^{s}_{\mathrm{d}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}((p,Q')^{s}_{\mathrm{d}}) \qquad \chi \in \mathsf{Strat}((p,Q')^{s}_{\mathrm{d}}, e')}{\chi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)}$$

$$\text{stable conj}\ \dfrac{Q \neq \varnothing \qquad \forall q \in Q.\ (p,Q)^{s}_{\mathrm{d}} \overset{u_q}{\rightarrowtail} [p,q]^{\wedge}_{\mathrm{a}} \ \wedge\ e_q = \mathsf{upd}(e,u_q) \in \mathsf{Win}_{\mathrm{a}}([p,q]^{\wedge}_{\mathrm{a}}) \ \wedge\ \psi_q \in \mathsf{Strat}([p,q]^{\wedge}_{\mathrm{a}}, e_q)}{\bigwedge(\{\neg\langle\tau\rangle\mathsf{T}\} \cup \{\psi_q \mid q \in Q\}) \in \mathsf{Strat}((p,Q)^{s}_{\mathrm{d}}, e)}$$

$$\text{stable finish}\ \dfrac{(p,\varnothing)^{s}_{\mathrm{d}} \overset{u}{\rightarrowtail} (p,\varnothing)_{\mathrm{d}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}((p,\varnothing)_{\mathrm{d}})}{\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \in \mathsf{Strat}((p,\varnothing)^{s}_{\mathrm{d}}, e)}$$

$$\text{branch}\ \dfrac{\begin{matrix}[p,Q]^{\varepsilon}_{\mathrm{a}} \overset{u}{\rightarrowtail} (p,\alpha,p',Q',Q_\alpha)^{\eta}_{\mathrm{d}} \\ e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}((p,\alpha,p',Q',Q_\alpha)^{\eta}_{\mathrm{d}}) \qquad \chi \in \mathsf{Strat}((p,\alpha,p',Q',Q_\alpha)^{\eta}_{\mathrm{d}}, e')\end{matrix}}{\chi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)}$$

$$\text{branch conj}\ \dfrac{\begin{matrix}g_{\mathrm{d}} = (p,\alpha,p',Q,Q_\alpha)^{\eta}_{\mathrm{d}} \overset{u_\alpha}{\rightarrowtail} [p',Q']^{\eta}_{\mathrm{a}} \overset{u'_\alpha}{\rightarrowtail} [p',Q']_{\mathrm{a}} \\ e_\alpha = \mathsf{upd}(\mathsf{upd}(e,u_\alpha),u'_\alpha) \in \mathsf{Win}_{\mathrm{a}}([p',Q']_{\mathrm{a}}) \qquad \varphi_\alpha \in \mathsf{Strat}([p',Q']_{\mathrm{a}}, e_\alpha) \\ \forall q \in Q.\ g_{\mathrm{d}} \overset{u_q}{\rightarrowtail} [p,q]^{\wedge}_{\mathrm{a}} \ \wedge\ e_q = \mathsf{upd}(e,u_q) \in \mathsf{Win}_{\mathrm{a}}([p,q]^{\wedge}_{\mathrm{a}}) \ \wedge\ \psi_q \in \mathsf{Strat}([p,q]^{\wedge}_{\mathrm{a}}, e_q)\end{matrix}}{\bigwedge(\{(\alpha)\varphi_\alpha\} \cup \{\psi_q \mid q \in Q\}) \in \mathsf{Strat}((p,\alpha,p',Q,Q_\alpha)^{\eta}_{\mathrm{d}}, e)}$$

Figure 8: Strategy formula constructions for Definition 5.1.

2. If $e \in \mathsf{Win}_\mathrm{a}([p,Q]^{\varepsilon}_{\mathrm{a}})$, then there is $\chi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)$ with $\mathsf{expr}^{\varepsilon}(\chi) \leq e$;
3. If $e \in \mathsf{Win}_\mathrm{a}([p,q]^{\wedge}_{\mathrm{a}})$, then there is $\psi \in \mathsf{Strat}([p,q]^{\wedge}_{\mathrm{a}}, e)$ with $\mathsf{expr}^{\wedge}(\psi) \leq e$.
4. If $e \in \mathsf{Win}_\mathrm{a}((p,Q)_{\mathrm{d}})$, then there is a standard conjunction $\bigwedge\Psi \in \mathsf{Strat}((p,Q)_{\mathrm{d}}, e)$ with $\mathsf{expr}^{\varepsilon}(\bigwedge\Psi) \leq e$;
5. If $e \in \mathsf{Win}_\mathrm{a}((p,Q)^{s}_{\mathrm{d}})$, then there is $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi \in \mathsf{Strat}((p,Q)^{s}_{\mathrm{d}}, e)$ with $\mathsf{expr}^{\varepsilon}(\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi) \leq e$;
6. If $e \in \mathsf{Win}_\mathrm{a}((p,\alpha,p',Q \setminus Q_\alpha, Q_\alpha)^{\eta}_{\mathrm{d}})$, then there is $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}((p,\alpha,p',Q \setminus Q_\alpha, Q_\alpha)^{\eta}_{\mathrm{d}}, e)$ with $\mathsf{expr}^{\varepsilon}(\bigwedge\{(\alpha)\varphi'\} \cup \Psi) \leq e$.

We apply induction over game positions g and energies e according to the inductive Definition 4.3. We distinguish cases depending on the kind of position.

1. Assume $e \in \mathsf{Win}_\mathrm{a}([p,Q]_{\mathrm{a}})$. This must be due to one of the following moves:

   Delay move $[p,Q]_{\mathrm{a}} \xrightarrow{\mathbf{0}} [p,Q_\varepsilon]^{\varepsilon}_{\mathrm{a}}$**:** We know that $e = \mathsf{upd}(e,\mathbf{0}) \in \mathsf{Win}_\mathrm{a}([p,Q_\varepsilon]^{\varepsilon}_{\mathrm{a}})$, so by induction hypothesis we know that there exists $\chi \in \mathsf{Strat}([p,Q_\varepsilon]^{\varepsilon}_{\mathrm{a}}, e)$ and $\mathsf{expr}^{\varepsilon}(\chi) \leq e$. But then $\langle\varepsilon\rangle\chi \in \mathsf{Strat}([p,Q]_{\mathrm{a}}, e)$ by rule (delay) of Definition 5.1 and $\mathsf{expr}(\langle\varepsilon\rangle\chi) = \mathsf{expr}^{\varepsilon}(\chi) \leq e$.

   Immediate conj. move $[p,Q]_{\mathrm{a}} \xrightarrow{-\hat{\mathbf{e}}_5} (p,Q)_{\mathrm{d}}$**:** It must hold that $e' = \mathsf{upd}(e,-\hat{\mathbf{e}}_5) \in \mathsf{Win}_\mathrm{a}((p,Q)_{\mathrm{d}})$, so by induction hypothesis we know that there exists a conjunction $\bigwedge\Psi \in \mathsf{Strat}((p,Q)_{\mathrm{d}}, e')$ and $\mathsf{expr}^{\varepsilon}(\bigwedge\Psi) \leq e'$. But then, $\bigwedge\Psi \in \mathsf{Strat}([p,Q]_{\mathrm{a}}, e)$ by rule (immediate conj) of Definition 5.1, and $\mathsf{expr}(\bigwedge\Psi) \leq e$.

2. Assume $e \in \mathsf{Win}_\mathrm{a}([p,Q]^{\varepsilon}_{\mathrm{a}})$. This must be due to one of the following moves:

   Procrastination move $[p,Q]^{\varepsilon}_{\mathrm{a}} \xrightarrow{\mathbf{0}} [p',Q]^{\varepsilon}_{\mathrm{a}}$**:** We know $\mathsf{upd}(e,\mathbf{0}) = e \in \mathsf{Win}_\mathrm{a}([p',Q]^{\varepsilon}_{\mathrm{a}})$. By induction hypothesis, there is $\chi \in \mathsf{Strat}([p',Q]^{\varepsilon}_{\mathrm{a}}, e)$ and $\mathsf{expr}^{\varepsilon}(\chi) \leq e$; therefore, by rule (procr) of Definition 5.1, $\chi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)$.

   Late (standard) conjunction move $[p,Q]^{\varepsilon}_{\mathrm{a}} \xrightarrow{\mathbf{0}} (p,Q)_{\mathrm{d}}$**:** It must be the case that $e \in \mathsf{Win}_\mathrm{a}((p,Q)_{\mathrm{d}})$. By induction hypothesis there is $\bigwedge\Psi \in \mathsf{Strat}((p,Q)_{\mathrm{d}}, e)$ and $\mathsf{expr}^{\varepsilon}(\bigwedge\Psi) \leq e$; therefore, by rule (late conj) of Definition 5.1, $\bigwedge\Psi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)$.

   Stable conjunction move $[p,Q]^{\varepsilon}_{\mathrm{a}} \xrightarrow{\mathbf{0}} (p,\{q \in Q \mid q \not\xrightarrow{\tau}\})^{s}_{\mathrm{d}}$**:** It must hold that p is stable and that $e \in \mathsf{Win}_\mathrm{a}((p,\{q \in Q \mid q \not\xrightarrow{\tau}\})^{s}_{\mathrm{d}})$. By induction hypothesis there is some formula $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi \in \mathsf{Strat}((p,\{q \in Q \mid q \not\xrightarrow{\tau}\})^{s}_{\mathrm{d}}, e)$ and $\mathsf{expr}^{\varepsilon}(\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi) \leq e$; thus, by rule (stable) of Definition 5.1, $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)$.

   Branch. conjunction move $[p,Q]^{\varepsilon}_{\mathrm{a}} \xrightarrow{\mathbf{0}} (p,\alpha,p',Q \setminus Q_\alpha, Q_\alpha)^{\eta}_{\mathrm{d}}$**:** It must hold that $e \in \mathsf{Win}_\mathrm{a}((p,\alpha,p', Q \setminus Q_\alpha, Q_\alpha)^{\eta}_{\mathrm{d}})$. By induction hypothesis there is a formula $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}((p,\alpha,p',Q \setminus Q_\alpha, Q_\alpha)^{\eta}_{\mathrm{d}}, e)$ and $\mathsf{expr}^{\varepsilon}(\bigwedge\{(\alpha)\varphi'\} \cup \Psi) \leq e$; therefore, by rule (branch) of Definition 5.1, $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}([p,Q]^{\varepsilon}_{\mathrm{a}})$.

3. Assume $e \in \mathsf{Win}_\mathrm{a}([p,q]^{\wedge}_{\mathrm{a}})$. This must be due to one of the following moves:

   Positive conjunct $[p,q]^{\wedge}_{\mathrm{a}} \xrightarrow{\mathtt{min}_{\{1,6\}},0,0,0,0,0,0,0} [p,\{q' \mid q \twoheadrightarrow q'\}]^{\varepsilon}_{\mathrm{a}}$**:** It must hold that $e' := \mathsf{upd}(e,(\mathtt{min}_{\{1,6\}}, 0,0,0,0,0,0,0)) \in \mathsf{Win}_\mathrm{a}([p,\{q' \mid q \twoheadrightarrow q'\}]^{\varepsilon}_{\mathrm{a}})$. By induction hypothesis there is some formula $\chi \in \mathsf{Strat}([p,\{q' \mid q \twoheadrightarrow q'\}]^{\varepsilon}_{\mathrm{a}}, e')$ and $\mathsf{expr}^{\varepsilon}(\chi) \leq e'$; therefore, by rule (pos) of Definition 5.1, $\langle\varepsilon\rangle\chi \in \mathsf{Strat}([p,q]^{\wedge}_{\mathrm{a}}, e)$.

   Negative conjunct $[p,q]^{\wedge}_{\mathrm{a}} \xrightarrow{\mathtt{min}_{\{1,7\}},0,0,0,0,0,0,-1} [q,\{p' \mid p \twoheadrightarrow p'\}]^{\varepsilon}_{\mathrm{a}}$**:** It holds that $e' := \mathsf{upd}(e,(\mathtt{min}_{\{1,7\}}, 0,0,0,0,0,0,-1)) \in \mathsf{Win}_\mathrm{a}([q,\{p' \mid p \twoheadrightarrow p'\}]^{\varepsilon}_{\mathrm{a}})$. By induction hypothesis there is some formula $\chi \in \mathsf{Strat}([q,\{p' \mid p \twoheadrightarrow p'\}]^{\varepsilon}_{\mathrm{a}}, e')$ and $\mathsf{expr}^{\varepsilon}(\chi) \leq e'$; therefore, by rule (neg) of Definition 5.1, $\neg\langle\varepsilon\rangle\chi \in \mathsf{Strat}([p,q]^{\wedge}_{\mathrm{a}}, e)$.

4. Assume $e \in \mathsf{Win}_a((p, Q)_d)$. For each move $(p, Q)_d \xrightarrow{-\hat{\mathbf{e}}_3} [p, q]^\wedge_a$, it must hold that $e' := \mathsf{upd}(e, -\hat{\mathbf{e}}_3) \in \mathsf{Win}_a([p, q]^\wedge_a)$, so by induction hypothesis there are $\psi_q \in \mathsf{Strat}([p, q]^\wedge_a)$ with $\mathsf{expr}^\wedge(\psi_q) \leq e'$. Therefore, by rule (conj) of Definition 5.1, $\bigwedge_{q \in Q} \psi_q \in \mathsf{Strat}((p, Q)_d, e)$, and $\mathsf{expr}^\varepsilon(\bigwedge_{q \in Q} \psi_q) = \hat{\mathbf{e}}_3 + \sup\{\mathsf{expr}^\wedge(\psi_q) \mid q \in Q\} \leq e$.

5. Assume $e \in \mathsf{Win}_a((p, Q)^s_d)$.

   Because of the move $(p, Q)^s_d \xrightarrow{-\hat{\mathbf{e}}_4 - \hat{\mathbf{e}}_8} (p, \varnothing)_d$, it must be the case that $\mathsf{upd}(e, -\hat{\mathbf{e}}_4 - \hat{\mathbf{e}}_8) \geq \mathbf{0}$, or equivalently, $e \geq \hat{\mathbf{e}}_4 + \hat{\mathbf{e}}_8$.

   If $Q = \varnothing$, we have $\mathsf{expr}^\varepsilon(\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\}) = \hat{\mathbf{e}}_4 + \hat{\mathbf{e}}_8 \leq e$ as required.

   If $Q \neq \varnothing$, we can argue similar to the previous case.

6. Assume $e \in \mathsf{Win}_a((p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^\eta_d)$. Then there are moves $(p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^\eta_d \xrightarrow{-\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3} [p, q]^\wedge_a$ for every $q \in Q \setminus Q_\alpha$; it must be the case that $e' := \mathsf{upd}(e, -\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3) \in \mathsf{Win}_a([p, q]^\wedge_a)$, so by induction hypothesis there are formulas $\psi_q \in \mathsf{Strat}([p, q]^\wedge_a)$ with $\mathsf{expr}^\wedge(\psi_q) \leq e'$. Also, for the moves $(p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^\eta_d \xrightarrow{\mathtt{min}_{\{1,6\}}, -1, -1, 0, 0, 0, 0, 0} [p', Q']^\eta_a \xrightarrow{-\hat{\mathbf{e}}_1} [p', Q']_a$, it must be the case that $e'' := \mathsf{upd}(\mathsf{upd}(e, (\mathtt{min}_{\{1,6\}}, -1, -1, 0, 0, 0, 0, 0)), -\hat{\mathbf{e}}_1) \in \mathsf{Win}_a([p', Q']_a)$, so by induction hypothesis there is some $\varphi' \in \mathsf{Strat}([p', Q']_a, e'')$ with $\mathsf{expr}(\varphi') \leq e''$.

   Hence, by rule (branch conj) of Definition 5.1, $\bigwedge\{(\alpha)\varphi'\} \cup \{\psi_q \mid q \in Q \setminus Q_\alpha\} \in \mathsf{Strat}((p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^\eta_d, e)$ and $\mathsf{expr}^\varepsilon(\bigwedge\{(\alpha)\varphi'\} \cup \{\psi_q \mid q \in Q \setminus Q_\alpha\}) \leq e$. □

Lemma 5.3. *If $\varphi \in \mathsf{Strat}([p, Q]_a, e)$, then φ distinguishes p from Q.*[9]

Proof. Again, to get an inductive property, we actually prove the following:

1. If $\varphi \in \mathsf{Strat}([p, Q]_a, e)$, then φ distinguishes p from Q;
2. If $\chi \in \mathsf{Strat}([p, Q]^\varepsilon_a, e)$ and $Q \twoheadrightarrow Q$, then $\langle\varepsilon\rangle\chi$ distinguishes p from Q;
3. If $\psi \in \mathsf{Strat}([p, q]^\wedge_a, e)$, then ψ distinguishes p from $\{q\}$.
4. If the standard conjunction $\bigwedge\Psi \in \mathsf{Strat}((p, Q)_d, e)$, then $\bigwedge\Psi$ distinguishes p from Q;
5. If $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi \in \mathsf{Strat}((p, Q)^s_d, e)$ and p is stable, then $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi$ distinguishes p from Q;
6. If $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}((p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^\eta_d, e)$, $p \xrightarrow{(\alpha)} p'$ and $Q_\alpha \subseteq Q$, then $\bigwedge\{(\alpha)\varphi'\} \cup \Psi$ distinguishes p from Q.

We prove the result by induction over the derivation of $\cdots \in \mathsf{Strat}(g, e)$ according to Definition 5.1.

1. Assume $\varphi \in \mathsf{Strat}([p, Q]_a, e)$.

   Due to rule (delay) in Definition 5.1: Then $\varphi = \langle\varepsilon\rangle\chi$ and for Q' with $Q \twoheadrightarrow Q'$ we have $\chi \in \mathsf{Strat}([p, Q']^\varepsilon_a, e)$. By induction hypothesis, $\langle\varepsilon\rangle\chi$ distinguishes p from Q', but then it also distinguishes p from $Q \subseteq Q'$.

   Due to rule (immediate conj) in Definition 5.1: The rule has premise $[p, Q]_a \xrightarrow{u} (p, Q)_d$, but this move can be a finishing move $[p, \varnothing]_a \xrightarrow{\mathbf{0}} (p, \varnothing)_d$ or an immediate conjunction move $[p, Q]_a \xrightarrow{-\hat{\mathbf{e}}_5} (p, Q)_d$ with $Q \neq \varnothing$. In either case, we have that $\varphi = \bigwedge\Psi \in \mathsf{Strat}((p, Q)_d, \mathsf{upd}(e, u))$. By induction hypothesis, $\bigwedge\Psi$ distinguishes p from Q, and this is exactly what we need to prove about $\varphi = \bigwedge\Psi$.

2. Assume $\chi \in \mathsf{Strat}([p, Q]^\varepsilon_a, e)$ and $Q \twoheadrightarrow Q$.

[9] lemma `Strategy_Formulas.lts_tau.strategy_formulas_distinguish`

Due to rule (procr) in Definition 5.1: Then there is a step $p \twoheadrightarrow p'$ such that $\chi \in \mathsf{Strat}([p', Q]^{\varepsilon}_{\mathrm{a}}, e)$. By induction hypothesis, we have that $\langle\varepsilon\rangle\chi$ distinguishes p' from Q, but then it also distinguishes p from Q.

Due to rule (observation) in Definition 5.1: Then $\chi = \langle a\rangle\varphi$ and there are $p \xrightarrow{a} p'$ and $Q \xrightarrow{a} Q'$ such that $\varphi \in \mathsf{Strat}([p', Q']_{\mathrm{a}}, \mathsf{upd}(e, -\hat{\mathbf{e}}_1))$. By induction hypothesis we have that φ distinguishes p' from Q'. Therefore, $p \in [\![\chi]\!] \subseteq [\![\langle\varepsilon\rangle\chi]\!]$. If there were some $q \in Q \cap [\![\langle\varepsilon\rangle\chi]\!]$, then we would have a path $q \twoheadrightarrow q' \xrightarrow{a} q'' \in [\![\varphi]\!]$. But $q' \in Q$ because $Q \twoheadrightarrow Q$ and therefore $q'' \in Q' \cap [\![\varphi]\!] = \varnothing$. Contradiction! Therefore $\langle\varepsilon\rangle\chi$ distinguishes p from Q.

Due to rule (late conj) in Definition 5.1: Then $\chi \in \mathsf{Strat}((p, Q)_{\mathrm{d}}, e)$. By induction hypothesis, χ distinguishes p from Q. As in the previous case, we use $Q \twoheadrightarrow Q$ to get that $\langle\varepsilon\rangle\chi$ distinguishes p from Q.

Due to rule (stable) in Definition 5.1: Then $\chi = \bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi \in \mathsf{Strat}((p, \{q \in Q \mid q \not\xrightarrow{\tau}\})^{s}_{\mathrm{d}}, e)$. By induction hypothesis, χ distinguishes p from the stable states in Q. Therefore, $p \in [\![\chi]\!] \subseteq [\![\langle\varepsilon\rangle\chi]\!]$. Unstable states do not satisfy $\neg\langle\tau\rangle\mathsf{T}$, so if there were some unstable $q \in Q \cap [\![\langle\varepsilon\rangle\chi]\!]$, then we would have a path $q \twoheadrightarrow q' \not\xrightarrow{\tau}$ with $q' \in [\![\chi]\!]$. But $q' \in Q$ because $Q \twoheadrightarrow Q$, so q' cannot satisfy χ by induction hypothesis. Contradiction! Therefore $\langle\varepsilon\rangle\chi$ distinguishes p from all states in Q.

Due to rule (branch) in Definition 5.1: Then $\chi \in \mathsf{Strat}((p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^{\eta}_{\mathrm{d}})$ (for some $p \xrightarrow{(\alpha)} p'$ and $Q_\alpha \subseteq Q$). By induction hypothesis, χ distinguishes p from Q. As in the previous case, we use $Q \twoheadrightarrow Q$ to get that $\langle\varepsilon\rangle\chi$ distinguishes p from Q.

3. Assume $\psi \in \mathsf{Strat}([p, q]^{\wedge}_{\mathrm{a}}, e)$.

Due to rule (pos) in Definition 5.1: Then ψ is of the form $\langle\varepsilon\rangle\chi$ and $\chi \in \mathsf{Strat}([p, Q']^{\varepsilon}_{\mathrm{a}}, \mathsf{upd}(e, (\mathtt{min}_{\{1,6\}}, 0, 0, 0, 0, 0, 0, 0)))$ for $\{q\} \twoheadrightarrow Q'$. By induction hypothesis, $\langle\varepsilon\rangle\chi$ distinguishes p from Q', and because $q \in Q'$, it also distinguishes p from q.

Due to rule (neg) in Definition 5.1: Then ψ is of the form $\neg\langle\varepsilon\rangle\chi$ and $\chi \in \mathsf{Strat}([q, P']^{\varepsilon}_{\mathrm{a}}, \mathsf{upd}(e, (\mathtt{min}_{\{1,7\}}, 0, 0, 0, 0, 0, 0, -1)))$ for $\{p\} \twoheadrightarrow P'$. By induction hypothesis, $\langle\varepsilon\rangle\chi$ distinguishes q from P', and because $p \in P'$, its negation ψ distinguishes p from q.

4. Assume $\bigwedge\Psi \in \mathsf{Strat}((p, Q)_{\mathrm{d}}, e)$. Note that $\bigwedge\Psi$ is a standard conjunction.

Due to rule (conj) in Definition 5.1: Then Ψ can be written as $\{\psi_q \mid q \in Q\}$, where each $\psi_q \in \mathsf{Strat}([p, q]^{\wedge}_{\mathrm{a}}, \mathsf{upd}(e, -\hat{\mathbf{e}}_3))$. By induction hypothesis, ψ_q distinguishes p from q, so also $\bigwedge\Psi$ distinguishes p from q. Because this holds for every $q \in Q$, we have that $\bigwedge\Psi$ distinguishes p from Q.

5. Assume $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi \in \mathsf{Strat}((p, Q)^{s}_{\mathrm{d}}, e)$ and p is stable.

Due to rule (stable conj) in Definition 5.1: Then Ψ can be written $\{\psi_q \mid q \in Q\}$, where $\psi_q \in \mathsf{Strat}([p, q]^{\wedge}_{\mathrm{a}}, \mathsf{upd}(e, -\hat{\mathbf{e}}_4))$. By induction hypothesis, ψ_q distinguishes p from q. Because this holds for every $q \in Q$ and p is stable, we have that $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi$ distinguishes p from Q.

Due to rule (stable fin.) in Definition 5.1: Then we must have $Q = \varnothing$ and $\Psi = \varnothing$. As p is stable, it satisfies $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\}$, i.e. the formula in $\mathsf{Strat}((p, Q)^{s}_{\mathrm{d}}, e)$.

6. Assume $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}((p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^{\eta}_{\mathrm{d}}, e)$, $p \xrightarrow{(\alpha)} p'$ and $Q_\alpha \subseteq Q$.

Due to rule (branch conj) in Definition 5.1: Then Ψ can be written as $\{\psi_q \mid q \in Q \setminus Q_\alpha\}$, where $\psi_q \in \mathsf{Strat}([p, q]^{\wedge}_{\mathrm{a}}, \mathsf{upd}(e, -\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3))$. By induction hypothesis, ψ_q distinguishes p from q. Because this holds for every $q \in Q \setminus Q_\alpha$, we have that $\bigwedge\Psi$ distinguishes p from $Q \setminus Q_\alpha$.
Moreover, there are moves $(p, \alpha, p', Q \setminus Q_\alpha, Q_\alpha)^{\eta}_{\mathrm{d}} \rightarrowtail [p', Q']^{\eta}_{\mathrm{a}} \rightarrowtail [p', Q']_{\mathrm{a}}$ where $p \xrightarrow{(\alpha)} p'$, $Q_\alpha \xrightarrow{(\alpha)} Q'$, and $\varphi' \in \mathsf{Strat}([p', Q']_{\mathrm{a}}, \mathsf{upd}(\mathsf{upd}(e, (\mathtt{min}_{\{1,6\}}, 0, 0, 0, 0, 0, 0, 0)), -\hat{\mathbf{e}}_1))$. By induction hypothesis, φ' distinguishes p' from Q', so $(\alpha)\varphi'$ distinguishes p from Q_α.
Together we have that $\bigwedge\{(\alpha)\varphi'\} \cup \Psi$ distinguishes p from Q. □

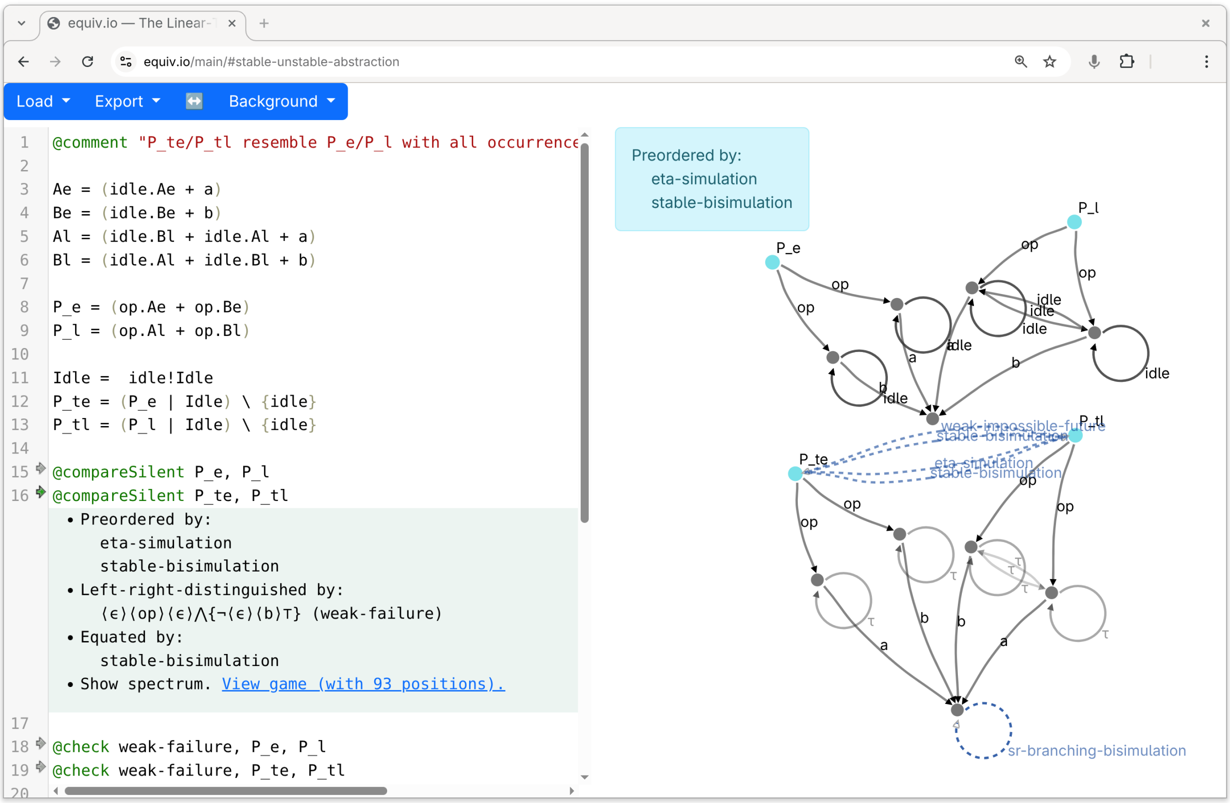

Figure 9: Screenshot of `equiv.io` solving Example 6.1.

6. Deciding All Weak Equivalences at Once, or Individually

The weak spectroscopy energy game enables algorithms to decide all considered behavioral equivalences. Our open-source prototype implementation can be tried out on `https://equiv.io`. Moreover, there is an extension of CAAL (Concurrency Workbench, Aalborg Edition, [24]) with the energy-game algorithm on `https://github.com/equivio/CAAL`. Both yield the expected output on van Glabbeek's finitary examples [6].

In this section, we briefly explicate how to check individual notions of equivalence and how to check all at once using our game.

6.1. Checking Individual Notions

Our generalized game characterization induces algorithms to *individually check notions of equivalence* on states of a transition system. The idea is to derive a (non-quantitative) reachability game, where positions are pairs of original positions and current energy levels.

Definition 6.1. Given a declining energy game $\mathcal{G} = (G, G_\mathrm{d}, \rightarrowtail, w)$, the *derived reachability game* $\mathcal{G}^R = (G^R, G^R_\mathrm{d}, \rightarrowtail_R)$ is played on pairs $G^R \coloneqq G \times (\mathbf{En}_\infty \cup \{\bot\})$ with $G^R_\mathrm{d} \coloneqq G_\mathrm{d} \times \mathbf{En}_\infty$. Lifted moves $(g, e) \rightarrowtail_R (g', e')$ are possible iff $e \neq \bot$ and if there is u such that $g \overset{u}{\rightarrowtail} g'$ and $e' = \mathsf{upd}(e, u)$, where $e' = \bot$ if $\mathsf{upd}(e, u)$ is undefined. (That is: positions with exhausted energies are stuck attacker positions.)

To check notion N, one explores the game $\mathcal{G}^R_\triangle$ starting from $([p, \{q\}]_\mathrm{a}, e_N)$ with the corresponding energy vector e_N from Figure 4. Positions where energy components become negative are won by the defender, and the exploration of the game-graph stops at such positions. If and only if the attacker wins from $([p, \{q\}]_\mathrm{a}, e_N)$, then $p \not\preceq_N q$.

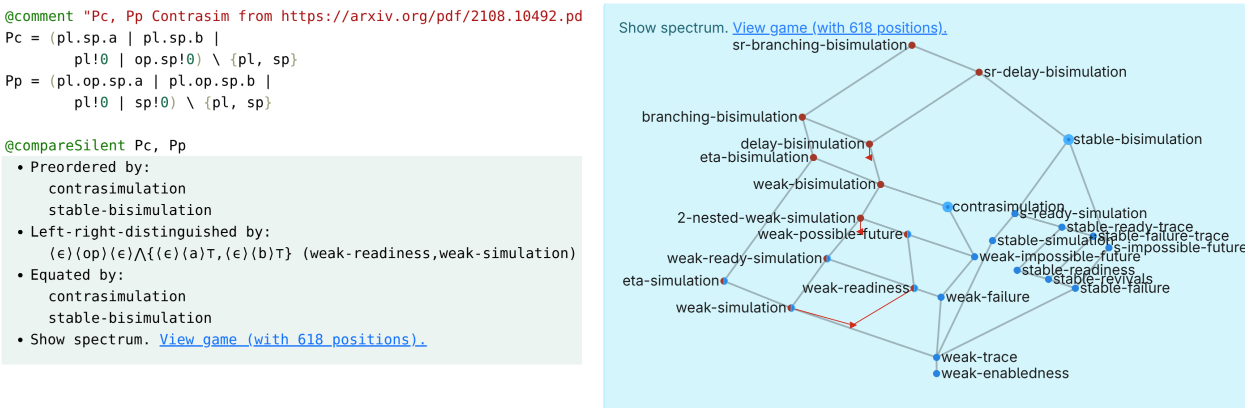

Figure 10: `equiv.io` showing the spectrum of equivalences (blue) and non-equivalences (red) between the processes mentioned at the beginning of Section 1 (cf. Example 6.2). Red triangles mark prices and directions of cheapest distinguishing formulas.

Depending on the selected energy vector, this will usually lead to a reachability game sized exponentially with respect to the transition system (due to the energy game sizes we will discuss in detail in the upcoming Section 7). As reachability games are simpler than energy games, the decision of the game winner itself can be done in linear time with respect to the game size. A winning attacker strategy can be seen as a construction of a distinguishing formula along the lines of Definition 5.1. Preferring clever moves speeds up the algorithm; for instance, if the budget allows arbitrarily many immediate conjunctions, prioritizing such moves can mitigate the exponential blowup due to nondeterminism in the transition system.

6.2. Checking All Notions at Once

More generally, one can *decide all equivalences at once* by computing the Pareto front of attacker-winning budgets $\mathsf{Win}_\mathrm{a}([p, \{q\}]_\mathrm{a})$. Details on how to do this can be found in Bisping [9]. Our implementation uses a verified variant of the algorithm with some slight improvements, presented in Lemke [23].

The output will inform about the most fitting preorders to relate and distinguish p and q, as in the following example:

Example 6.1. Let us try our initial Example 2.1 of abstracted processes (Figure 9 and `https://equiv.io/#stable-unstable-abstraction`). The browser tool takes about 100 ms (considering a game of 112 positions) to report that $\mathsf{P_e}$ and P_ℓ are stable *and* unstable readiness-equivalent. $\mathsf{P}^\tau_\mathsf{e}$ and P^τ_ℓ on the other hand are stable-bisimilar. This output immediately tells us that only notions either strictly finer than readiness or coarser than stable bisimilarity can be congruences for τ-abstraction. In particular, (unstable) weak failures, which Gazda et al. [13, Corr. 9] report to be a congruence for abstraction, cannot be one because the failure formula $\langle\varepsilon\rangle\langle op\rangle\langle\varepsilon\rangle\bigwedge\{\neg\langle\varepsilon\rangle\langle a\rangle\top\}$ distinguishes $\mathsf{P}^\tau_\mathsf{e}$ from P^τ_ℓ, analogously to φ_τ of Example 2.2.

Example 6.2. In the introduction, we have cited Bell's example processes $\mathsf{P_c}$ and $\mathsf{P_p}$ [2]. We can quickly input them in CCS notation and ask our tool to tell us about the most fitting equivalences (available at `https://equiv.io/#weak-sims`). The spectrum output (Figure 10) shows that all equivalences below contrasimilarity and stable bisimilarity relate the two processes. This aligns with what Bisping and Montanari [21] report.

6.3. "All" Weak Equivalences?

Of course, our treatment only addresses all the notions in the spectrum of Figure 4, not *all* weak notions that have ever been conceived. This is in part intentional to limit the bloat of spectrum and game. Let us still briefly discuss how some additional notions could be covered.

Revivals and Decorated Traces. The notions of stable revivals, failure traces, and ready traces are relevant to the CSP community [25]. Their modal characterization demands to differentiate between the modal depth of the deepest positive conjunct and of the other positive conjuncts. For instance, ready traces may have one arbitrarily deep positive conjunct at each conjunction, but all the others are capped to modal depth 1, expressing which other actions would be enabled or disabled at this point of a trace. Thus, these equivalences need an additional dimension for `expr`-measurements.

Bisping [9] shows how to include this aspect into the expressiveness metric and the game for notions that have no special treatment of internal behavior. This comes at the cost of making conjunctions more complicated, but can be transferred to the present game. In fact, the implementation on `https://equiv.io` does employ this trick, thereby actually using a 9-dimensional game with richer stable conjunctions.

Divergence and Completed Observations. Logic and game, as we have presented them, are blind to divergence and to completed observations. Van Glabbeek [6] uses additional modalities: Δ for divergence with $[\![\Delta]\!] := \{p \mid p \xrightarrow{\tau}{}^{\omega}\}$; 0 for completed observations with $[\![0]\!] := \{p \mid \forall a.p \not\xrightarrow{a}\}$ (and $\lambda := 0 \vee \Delta$).

We have decided against including these in the game. At least on finite-state systems, they may be understood to be special action observations. In this view, divergence and completion are something to be modelled (or added through pre-processing) into a system $\mathcal{S}$ before turning to our game of equivalence questions on $\mathcal{S}'$.

For 0, the transformation from $\mathcal{S}$ to $\mathcal{S}'$ is obvious: Add a $p \xrightarrow{\checkmark} \bot$ to the transition system for each $p \in \mathcal{P}$ where $p \not\xrightarrow{a}$ for every $a \in \Sigma \setminus \{\tau\}$ (with $\checkmark$ and $\bot$ fresh). Then $[\![\langle\checkmark\rangle\top]\!]^{\mathcal{S}'} = [\![0]\!]^{\mathcal{S}}$.

For finite-state systems, divergence can be addressed by an argument from Groote et al. [26]. Add a state $\bot$, an action $\delta \notin \Sigma$ and transitions $p \xrightarrow{\delta} \bot$ to the transition system for each $p \in \mathcal{P}$ that lives on a τ-cycle $p \xrightarrow{\tau}{}^{+} p$. Then $[\![\langle\varepsilon\rangle\langle\delta\rangle\top]\!]^{\mathcal{S}'} = [\![\Delta]\!]^{\mathcal{S}}$. Fokkink et al. [19] define a unary *diverges-while* operator $\Delta\varphi$ to characterize divergence-preserving branching bisimilarity; this operator is then naturally expressed as branching conjunction $\langle\varepsilon\rangle\bigwedge\{\langle\delta\rangle\top, \varphi\}$.

For infinite systems, divergence is more tricky. Just like infinite traces, it depends on the possibility of characterizing infinite-duration attacks. On the game level, for infinite plays to be winnable by the attacker, the game must have a Büchi winning condition. Such a richer game model is the route taken by de Frutos Escrig et al. [14] to characterize various divergence-aware bisimilarities. For our setting aimed at algorithms on finite-state systems, this would seem overkill.

7. Complexity and How to Decrease It

Our approach to decide all behavioral equivalences at once takes exponential time and space as the subset constructions make the spectroscopy game exponential. Because many of the individual equivalence problems in the linear-time–branching-time spectrum already have exponential complexity on their own, there cannot be sub-exponential solutions to the spectroscopy problem. Still, we want to try to achieve some level of applicability.

In this section, we give detailed bounds for the spectroscopy game $\mathcal{G}_\vartriangle$, show how game size can be decreased substantially, and prove relaxed correctness of a simplified spectroscopy game.

7.1. Complexity of the Weak Spectroscopy

Our implementation uses Lemke's [23] verified algorithm to determine Pareto fronts for multi-weighted energy games. The algorithm takes $\mathcal{O}(o_{\rightarrowtail} \cdot |G|^{2N} \cdot (N^2 + |G|^{N-1} \cdot N))$ time and $\mathcal{O}(|G|^N \cdot N)$ space for an N-dimensional declining energy game $(G, G_{\mathrm{d}}, \rightarrowtail, w)$, where $o_{\rightarrowtail}$ denotes the out-degree of $\rightarrowtail$. For this paper's weak $N = 8$-dimensional spectroscopy game, $\mathcal{G}_\vartriangle$, we have $|G_\vartriangle| \in \mathcal{O}(|{\rightarrow}| \cdot 3^{|\mathcal{P}|})$ and also $o_\vartriangle \in \mathcal{O}(|{\rightarrow}| \cdot 2^{|\mathcal{P}|})$, because of the defender branching positions and their surroundings. This amounts to exponential time complexity of $\mathcal{O}(|{\rightarrow}| \cdot 2^{|\mathcal{P}|} \cdot (|{\rightarrow}| \cdot 3^{|\mathcal{P}|})^{16} \cdot (|{\rightarrow}| \cdot 3^{|\mathcal{P}|})^{7}) = \mathcal{O}(|{\rightarrow}|^{24} \cdot 2^{|\mathcal{P}|} \cdot 3^{23|\mathcal{P}|})$, and space complexity $\mathcal{O}(|{\rightarrow}|^{8} \cdot 3^{8|\mathcal{P}|})$. Of course, much of the big numbers caused by the dimensionality do not materialize as the dimensions are partially linked; and the main exponentiality happens only due to nondeterminism.

The exponential out-degree is due to branching conjunction moves. That these would need exponentially many outgoing moves seems off: These moves are for η- and branching bisimilarity, which are known to

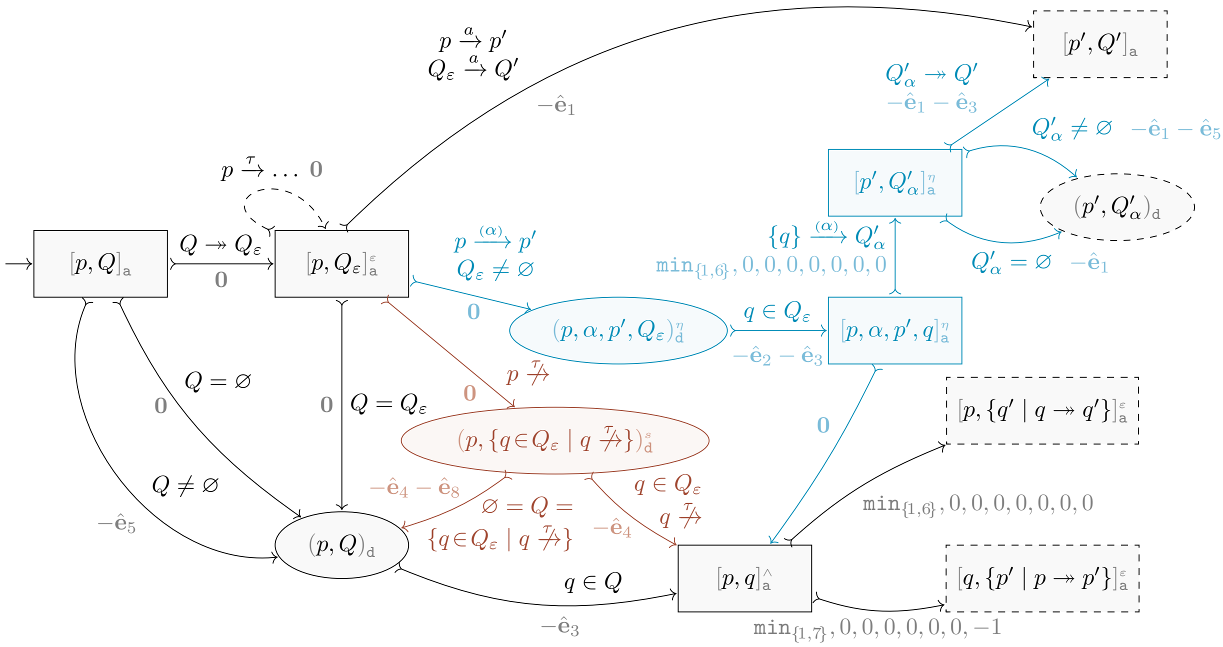

Figure 11: Schematic spectroscopy game $\mathcal{G}_\blacktriangle$ with the blue part updated by Definition 7.1.

be at the less expensive (sub-cubic [26]) end of equivalence problems in the spectrum. De Frutos Escrig et al.'s branching bisimulation game [14] is polynomially sized. Thus, a derived reachability game of the spectroscopy game for branching bisimilarity should too be polynomial in size if we apply clever optimizations as hinted to at the end of Subsection 6.1. The next subsection will show how to simplify the spectroscopy game to achieve this reduction of size around branching conjunctions.

7.2. Improving the Complexity

To reduce the out-degree to be linear, let us reformulate the branching conjunction part of the game to be closer to the operational definitions of η- and branching bisimilarity (Definition 2.4). We can still solve the main spectroscopy problem, but will lose some resolution about the number of nested conjunctions.

If we read Definition 2.4 directly as a game, it differs from the branching conjunction moves in Definition 4.6, because the latter require the attacker to name as Q_α ex-ante which q' to challenge directly and which ones only after the α step, and to have one continuation for the whole Q_α group. Let us rephrase this part to match Definition 2.4:

Definition 7.1. The *simplified spectroscopy energy game* $\mathcal{G}^\mathcal{S}_\blacktriangle$ extends Definition 4.5 by

- *defender branching positions* $(p, \alpha, p', Q)^\eta_\mathrm{d} \in G_\mathrm{d}$,
- *attacker branching clause positions* $[p, \alpha, p', q]^\eta_\mathrm{a} \in G_\mathrm{a}$,
- *attacker branching positions* $[p, Q]^\eta_\mathrm{a} \in G_\mathrm{a}$,

where $p, p' \in \mathcal{P}$, $Q \in \mathbf{2}^\mathcal{P}$ and $\alpha \in \Sigma$, and seven kinds of moves:

- *branching conjunction* $[p, Q]^\varepsilon_\mathrm{a} \xrightarrow{0,0,0,0,0,0,0,0} (p, \alpha, p', Q)^\eta_\mathrm{d}$ if $p \xrightarrow{(\alpha)} p'$,$Q \neq \varnothing$
- *branching answer* $(p, \alpha, p', Q)^\eta_\mathrm{d} \xrightarrow{0,-1,-1,0,0,0,0,0} [p, \alpha, p', q]^\eta_\mathrm{a}$ if $q \in Q$,
- *branching observation* $[p, \alpha, p', q]^\eta_\mathrm{a} \xrightarrow{\mathtt{min}_{\{1,6\}},0,0,0,0,0,0,0} [p', Q']^\eta_\mathrm{a}$ if $\{q\} \xrightarrow{(\alpha)} Q'$,

$$\text{branch}\ \frac{[p,Q]^{\varepsilon}_{\mathrm{a}} \overset{u}{\rightarrowtail} (p,\alpha,p',Q)^{\eta}_{\mathrm{d}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}((p,\alpha,p',Q)^{\eta}_{\mathrm{d}}) \qquad \chi \in \mathsf{Strat}_{\blacktriangle}((p,\alpha,p',Q)^{\eta}_{\mathrm{d}}, e')}{\chi \in \mathsf{Strat}_{\blacktriangle}([p,Q]^{\varepsilon}_{\mathrm{a}}, e)}$$

$$\text{branch immediate acct}\ \frac{[p,Q]^{\eta}_{\mathrm{a}} \overset{u}{\rightarrowtail} (p,Q)_{\mathrm{d}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}((p,Q)_{\mathrm{d}}) \qquad \varphi \in \mathsf{Strat}_{\blacktriangle}((p,Q)_{\mathrm{d}}, e')}{\varphi \in \mathsf{Strat}_{\blacktriangle}([p,Q]^{\eta}_{\mathrm{a}}, e)}$$

$$\text{branch late acct}\ \frac{[p,Q]^{\eta}_{\mathrm{a}} \overset{u}{\rightarrowtail} [p,Q']^{\varepsilon}_{\mathrm{a}} \qquad e' = \mathsf{upd}(e,u) \in \mathsf{Win}_{\mathrm{a}}([p,Q']^{\varepsilon}_{\mathrm{a}}) \qquad \chi \in \mathsf{Strat}_{\blacktriangle}([p,Q']^{\varepsilon}_{\mathrm{a}}, e')}{\langle\varepsilon\rangle\chi \in \mathsf{Strat}_{\blacktriangle}([p,Q]^{\eta}_{\mathrm{a}}, e)}$$

$$\text{branch conj}\ \frac{\begin{array}{c}\forall q \in Q.\ (p,\alpha,p',Q)^{\eta}_{\mathrm{d}} \overset{u}{\rightarrowtail} [p,\alpha,p',q]^{\eta}_{\mathrm{a}} \overset{u'}{\rightarrowtail} g_q \\ \wedge\ e_q = \mathsf{upd}(\mathsf{upd}(e,u),u') \in \mathsf{Win}_{\mathrm{a}}(g_q)\ \wedge\ \psi_q \in \mathsf{Strat}_{\blacktriangle}(g_q, e_q) \\ Q_\alpha = \{q_\alpha \in Q \mid \exists Q'.\, g_{q_\alpha} = [p',Q']^{\eta}_{\mathrm{a}}\} \qquad \varphi_\alpha = \mathrm{merge}(\{\psi_{q_\alpha} \mid q_\alpha \in Q_\alpha\})\end{array}}{\bigwedge(\{(\alpha)\varphi_\alpha\} \cup \{\psi_q \mid q \in Q \setminus Q_\alpha\}) \in \mathsf{Strat}_{\blacktriangle}((p,\alpha,p',Q)^{\eta}_{\mathrm{d}}, e)}$$

where $\mathrm{merge}(\Phi) := \begin{cases} \langle\varepsilon\rangle\bigwedge\mathrm{flatten}(\Phi) & \text{if each } \varphi \in \Phi \setminus \{\mathsf{T}\} \text{ starts with } \langle\varepsilon\rangle \\ \bigwedge\mathrm{flatten}(\Phi) & \text{otherwise} \end{cases}$

and $\mathrm{flatten}(\Phi) := \bigcup_{\varphi\in\Phi} \begin{cases} \Psi & \text{if } \varphi = \bigwedge\Psi \\ \{\varphi\} & \text{if } \varphi = \langle\varepsilon\rangle\chi \end{cases}$.

Figure 12: Rules that replace (branch) and (branch conj) for Definition 7.2.

- *branching reset* $[p,\alpha,p',q]^{\eta}_{\mathrm{a}} \xrightarrow{0,0,0,0,0,0,0,0} [p,q]^{\wedge}_{\mathrm{a}}$,
- *immediate branching accounting* $[p,Q]^{\eta}_{\mathrm{a}} \xrightarrow{-1,0,0,0,-1,0,0,0} (p,Q)_{\mathrm{d}}$ if $Q \neq \varnothing$,
- *immediate branching finishing* $[p,\varnothing]^{\eta}_{\mathrm{a}} \xrightarrow{-1,0,0,0,0,0,0,0} (p,\varnothing)_{\mathrm{d}}$,
- *late branching accounting* $[p,Q]^{\eta}_{\mathrm{a}} \xrightarrow{-1,0,-1,0,0,0,0,0} [p,Q']^{\varepsilon}_{\mathrm{a}}$ if $Q \twoheadrightarrow Q'$.

Figure 11 gives a schematic representation. (Only the steel-blue part for branching conjunctions has changed.) Note that the immediate branching accounting/finishing moves lead to defender conjunction positions, while the late branching accounting move skips it, but still charges for the conjunction by an additional energy update of $-\hat{\mathbf{e}}_3$.

The simplified game part encodes nested conjunctions of the form $\bigwedge\{(\alpha)\bigwedge\Psi', \psi_1, \ldots\}$ or the cheaper form $\bigwedge\{(\alpha)\langle\varepsilon\rangle\bigwedge\Psi', \psi_1, \ldots\}$. The Ψ' are the formulas that matter after branching observation moves, while the ψ_i come from the resets. More formally:

Definition 7.2 (Formulas with simplified branching conjunction)**.** For strategy formulas $\mathsf{Strat}_{\blacktriangle}$ on the simplified spectroscopy game, we replace the rules (branch) and (branch conj) of Definition 5.1 by the ones in Figure 12. We copy the other rules from Figure 8 with a formal replacement of Strat by $\mathsf{Strat}_{\blacktriangle}$.

The simplified game makes a huge difference in practice: Using it, the browser tool of Section 6 can conduct a spectroscopy on examples like Peterson's mutual exclusion protocol and specification in 200 milliseconds, whereas it runs out of memory if it tries to construct the full unsimplified game.[10]

Complexitywise, for the simplified game, $\mathcal{G}_{\blacktriangle}$, we have just $|G_{\blacktriangle}| \in \mathcal{O}(|{\rightarrow}| \cdot 2^{|\mathcal{P}|})$ and also $o_{\blacktriangle} \in \mathcal{O}(|{\rightarrow}|)$. Deciding the whole game still has exponential time complexity of $\mathcal{O}(|{\rightarrow}| \cdot (|{\rightarrow}| \cdot 2^{|\mathcal{P}|})^{16} \cdot (|{\rightarrow}| \cdot 2^{|\mathcal{P}|})^{7}) = \mathcal{O}(|{\rightarrow}|^{24} \cdot 2^{23|\mathcal{P}|})$, and space complexity $\mathcal{O}(|{\rightarrow}|^{8} \cdot 2^{8|\mathcal{P}|})$, but these are much lower bounds than in the original game.

Bisping [9] describes a trick to cap the energy lattice to $\{0,1,\infty\}^8$. Thereby, the size of possible Pareto fronts is bounded, and thus decoupled from the game size. This further improves space complexity to $\mathcal{O}(|{\rightarrow}| \cdot 2^{|\mathcal{P}|})$ and overall time complexity to $\mathcal{O}(|{\rightarrow}| \cdot (|{\rightarrow}| \cdot 2^{|\mathcal{P}|})^{16}) = \mathcal{O}(|{\rightarrow}|^{17} \cdot 2^{16|\mathcal{P}|})$.

[10] The model can be found on `https://equiv.io/#peterson-mutex-silent`.

7.3. Correctness of the Simplified Spectroscopy Game

As the counting of conjunctions in the simplified game does not fully align with our theorems of Section 5, we have to relax our correctness claims. Sometimes the inner conjunction $\bigwedge\Psi'$ might be unifiable to contain only one element, such that a minimal-energy attack formula needs one fewer conjunction level than what the simplified game predicts. Lemma 5.1 would not hold for this formulation.

This is a rather theoretical problem: Notions that allow branching/η observations do not care about precise counting of conjunctions; either there are immediate conjunctions or there are none. Therefore, an implementation not caring about the very-cheapest formulas in the sense of Section 3, and only interested in notions where $(e_N)_2 > 0$ implies $(e_N)_3, (e_N)_5 \in \{0, \infty\}$, can employ the simpler formulation of Definition 7.1.

If we simplify the lattice of energies to *not count the conjunctions whenever there are branching conjunctions,* or round up the number of conjunctions to ∞ whenever it is nonzero, we can prove that the simpler game is equally correct, as follows.

Definition 7.3 (Simplified energies and formula prices)**.** Let $\mathbf{En}_{\infty\blacktriangle}$ be the union of $\mathbb{N} \times \{0\} \times \mathbb{N}^6$ and $\mathbb{N} \times \{\infty\} \times \{0, \infty\} \times \mathbb{N} \times \{0, \infty\} \times \mathbb{N}^3$.

We form simplified expressiveness prices by rounding up the prices of Definition 3.3: $\mathsf{expr}_\blacktriangle(\varphi) = \min\{e \in \mathbf{En}_{\infty\blacktriangle} \mid e \geq \mathsf{expr}(\varphi)\}$ (and similarly $\mathsf{expr}^\varepsilon_\blacktriangle$ and $\mathsf{expr}^\wedge_\blacktriangle$).

The first subset of $\mathbf{En}_{\infty\blacktriangle}$ contains energy values for plays without branching conjunction, and everything else is counted normally. The second subset contains energy values for plays with branching conjunction; then we only distinguish whether branching, standard or immediate conjunctions are absent (the corresponding dimension is 0) or at least one conjunction is present (the corresponding dimension is rounded up to ∞). Stable conjunctions can still be counted exactly.

Energy updates (Definition 4.1) are not changed; let us only recall that when applying a relative update -1, we set $\infty - 1 = \infty$ (as in the third dimension in Example 4.1).

The definition of winning budgets (Definition 4.3) does not change either; but let us recall that, as sometimes finite values for dimensions are missing, the concrete *minimum* values in $\mathbf{En}_{\infty\blacktriangle}$ for $\mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}(g)$ contain infinite components more often.

Further definitions of the declining energy game etc. are all analogous to the above. Then we can prove correctness on simplified game $\mathcal{G}_\blacktriangle$ and energies $\mathbf{En}_{\infty\blacktriangle}$ corresponding to Theorem 5.1:

Theorem 7.1. *For all $e \in \mathbf{En}_{\infty\blacktriangle}$, $p \in \mathcal{P}$, $Q \in \mathbf{2}^\mathcal{P}$, the following are equivalent:*

1. *There exists a formula $\varphi \in \mathsf{HML}_\mathrm{srbb}$ with price $\mathsf{expr}_\blacktriangle(\varphi) \leq e$ that distinguishes p from Q.*
2. *Attacker wins $\mathcal{G}^\mathcal{S}_\blacktriangle$ from $[p, Q]_\mathrm{a}$ with e (that is, $e \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}^\mathcal{S}_\blacktriangle}([p, Q]_\mathrm{a})$).*

The proof of the theorem is given through the following three lemmas.

Lemma 7.1. *If $\varphi \in \mathsf{HML}_\mathrm{srbb}$ distinguishes p from Q, then* $\mathsf{expr}_\blacktriangle(\varphi) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, Q]_\mathrm{a})$.

Proof. Similarly to Lemma 5.1, we prove the following inductive property:

1. If $\varphi \in \mathsf{HML}_\mathrm{srbb}$ distinguishes p from $Q \neq \varnothing$, then $\mathsf{expr}_\blacktriangle(\varphi) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, Q]_\mathrm{a})$;

2$_\blacktriangle$. If χ distinguishes p from $Q \neq \varnothing$ and Q is closed under $\twoheadrightarrow$ (that is $Q \twoheadrightarrow Q$), then $\mathsf{expr}^\varepsilon_\blacktriangle(\chi) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, Q]^\varepsilon_\mathrm{a})$;

3. If ψ distinguishes p from q, then $\mathsf{expr}^\wedge_\blacktriangle(\psi) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, q]^\wedge_\mathrm{a})$.

4. If $\bigwedge\Psi$ distinguishes p from $Q \neq \varnothing$, then $\mathsf{expr}^\varepsilon_\blacktriangle(\bigwedge\Psi) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}((p, Q)_\mathrm{d})$;

5. If $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi$ distinguishes p from $Q \neq \varnothing$ and the processes in Q are stable, then $\mathsf{expr}^\varepsilon_\blacktriangle(\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}((p, Q)^\mathrm{s}_\mathrm{d})$;

6$_\blacktriangle$. If $\bigwedge\{(\alpha)\varphi'\} \cup \Psi$ distinguishes p from Q, then, for any $p \xrightarrow{(\alpha)} p' \in [\![\varphi']\!]$, $\mathsf{expr}^\varepsilon_\blacktriangle(\bigwedge\{(\alpha)\varphi'\} \cup \Psi) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}((p, \alpha, p', Q)^\eta_\mathrm{d})$.

We redo just the proofs of the clauses that really change:

$2_\blacktriangle$. Assume χ distinguishes p from $Q \neq \varnothing$ (and $Q \twoheadrightarrow Q$).

$\chi = \langle a \rangle \varphi'$, $\chi = \bigwedge \Psi$, or $\chi = \bigwedge \{\neg \langle \tau \rangle \mathsf{T}\} \cup \Psi$: Same as in Lemma 5.1.

$\chi = \bigwedge \{(\alpha)\varphi'\} \cup \Psi$: Note that there must exist $p \xrightarrow{(\alpha)} p' \in [\![\varphi']\!]$ (otherwise $p \notin [\![(\alpha)\varphi']\!] \supseteq [\![\chi]\!]$, so χ would not distinguish p from anything). Pick such a p'. Then there is the move $[p, Q]^{\varepsilon}_{\mathrm{a}} \xrightarrow{\text{branching conj.}} (p, \alpha, p', Q)^{\eta}_{\mathrm{d}}$; so we can use the proof for $(p, \alpha, p', Q)^{\eta}_{\mathrm{d}}$ that follows in (case $6_\blacktriangle$) and Definition 4.3 to get $\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\chi) \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}([p, Q]^{\varepsilon}_{\mathrm{a}})$.

$6_\blacktriangle$. Assume $\bigwedge \{(\alpha)\varphi'\} \cup \Psi$ distinguishes p from Q and $p \xrightarrow{(\alpha)} p' \in [\![\varphi']\!]$.

There are the moves $(p, \alpha, p', Q)^{\eta}_{\mathrm{d}} \xrightarrow{\text{br. answer}} [p, \alpha, p', q]^{\eta}_{\mathrm{a}}$ for all $q \in Q$. Note that the branching answer move, although it has a nonzero price, does not update an expressiveness in $\mathbf{En}_{\infty\blacktriangle}$, as it only changes dimensions that are ∞ anyway: $\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \{(\alpha)\varphi'\} \cup \Psi) = \mathsf{upd}(\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \{(\alpha)\varphi'\} \cup \Psi), (0, -1, -1, 0, 0, 0, 0, 0))$.

For every such position we have to prove that $\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \{(\alpha)\varphi'\} \cup \Psi) \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}([p, \alpha, p', q]^{\eta}_{\mathrm{a}})$. We distinguish whether $q \in [\![(\alpha)\varphi']\!]$ or not.

If $q \in [\![(\alpha)\varphi']\!]$, the attacker continues with $[p, \alpha, p', q]^{\eta}_{\mathrm{a}} \xrightarrow{\text{br. reset}} [p, q]^{\wedge}_{\mathrm{a}}$. As $\bigwedge \{(\alpha)\varphi'\} \cup \Psi$ distinguishes p from q, it must be the case that one of the conjuncts, say $\psi_q \in \Psi$, does not hold in q; in other words, ψ_q distinguishes p from q. Using induction hypothesis (case 3), we conclude that $\mathsf{expr}^{\wedge}_{\blacktriangle}(\psi_q) \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}([p, q]^{\wedge}_{\mathrm{a}})$ and because $\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \{(\alpha)\varphi'\} \cup \Psi)$ is larger, it is also in $\mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}([p, q]^{\wedge}_{\mathrm{a}})$.

If, however, $q \notin [\![(\alpha)\varphi']\!]$, the attacker chooses to continue with $[p, \alpha, p', q]^{\eta}_{\mathrm{a}} \xrightarrow{\text{br. observation}} [p', Q']^{\eta}_{\mathrm{a}}$ for $\{q\} \xrightarrow{(\alpha)} Q'$. Using Definition 3.3, we can find that $\mathsf{upd}(\mathsf{upd}(\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \{(\alpha)\varphi'\} \cup \Psi), (\mathtt{min}_{\{1,6\}}, 0, 0, 0, 0, 0, 0, 0)), -\hat{\mathbf{e}}_1) \geq \mathsf{expr}_{\blacktriangle}(\varphi')$.

We know that φ' distinguishes p' from Q'; the attacker just continues depending on which move is the most advantageous for them. If $Q' = \varnothing$, the best move is $[p', \varnothing]^{\eta}_{\mathrm{a}} \xrightarrow{\text{immediate br. finishing}} (p', \varnothing)_{\mathrm{d}}$; then one can finish the proof using $\mathsf{expr}_{\blacktriangle}(\varphi') \geq \mathbf{0} \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}((p', \varnothing)_{\mathrm{d}})$.

If φ' is an immediate conjunction and $Q' \neq \varnothing$ (then we must have that $\varphi' \neq \mathsf{T}$), the attacker continues with $[p', Q']^{\eta}_{\mathrm{a}} \xrightarrow{\text{immediate br. acct.}} (p', Q')_{\mathrm{d}}$, and the continuation can be handled as in (case 1). (The additional update by $-\hat{\mathbf{e}}_5$ is absorbed because $(\mathsf{expr}_{\blacktriangle}(\varphi'))_5 = \infty$.)

Otherwise, $\varphi' = \langle \varepsilon \rangle \chi'$, and the attacker continues with $[p', Q']^{\eta}_{\mathrm{a}} \xrightarrow{\text{late br. acct.}} (p', Q'')_{\mathrm{d}}$ for $Q' \twoheadrightarrow Q''$. (There is an additional update by $-\hat{\mathbf{e}}_3$, but that does not matter, because $(\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \{(\alpha)\varphi'\} \cup \Psi))_3 = \infty$.)

If χ' is a standard conjunction, the continuation can be handled as in (case $2_\blacktriangle$) directly. Otherwise, the continuation can be handled as if χ' were the standard conjunction $\bigwedge \{\langle \varepsilon \rangle \chi'\}$, using induction hypothesis (case 3); the added conjunction has no influence on the expressiveness price $\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \{(\alpha)\varphi'\} \cup \Psi) = \mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \{(\alpha)\langle \varepsilon \rangle \bigwedge \{\varphi'\}\} \cup \Psi)$, as the third dimension is ∞. □

Lemma 7.2. *If $e \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}([p, Q]_{\mathrm{a}})$, then there is $\varphi \in \mathsf{Strat}_{\blacktriangle}([p, Q]_{\mathrm{a}}, e)$ with $\mathsf{expr}_{\blacktriangle}(\varphi) \leq e$.*

Proof. We prove a more detailed result, namely:

1. If $e \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}([p, Q]_{\mathrm{a}})$, then there is $\varphi \in \mathsf{Strat}_{\blacktriangle}([p, Q]_{\mathrm{a}}, e)$ with price $\mathsf{expr}_{\blacktriangle}(\varphi) \leq e$;

$2_\blacktriangle$. If $e \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}([p, Q]^{\varepsilon}_{\mathrm{a}})$, then there is $\chi \in \mathsf{Strat}_{\blacktriangle}([p, Q]^{\varepsilon}_{\mathrm{a}}, e)$ with price $\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\chi) \leq e$;

3. If $e \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}([p, q]^{\wedge}_{\mathrm{a}})$, then there is $\psi \in \mathsf{Strat}_{\blacktriangle}([p, q]^{\wedge}_{\mathrm{a}}, e)$ with price $\mathsf{expr}^{\wedge}_{\blacktriangle}(\psi) \leq e$.

4. If $e \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}((p, Q)_{\mathrm{d}})$, then there is $\bigwedge \Psi \in \mathsf{Strat}_{\blacktriangle}((p, Q)_{\mathrm{d}})$ with price $\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \Psi) \leq e$;

5. If $e \in \mathsf{Win}^{\mathcal{G}_\blacktriangle}_{\mathrm{a}}((p, Q)^{s}_{\mathrm{d}})$, then there is $\bigwedge \{\neg \langle \tau \rangle \mathsf{T}\} \cup \Psi \in \mathsf{Strat}_{\blacktriangle}((p, Q)^{s}_{\mathrm{d}})$ with price $\mathsf{expr}^{\varepsilon}_{\blacktriangle}(\bigwedge \{\neg \langle \tau \rangle \mathsf{T}\} \cup \Psi) \leq e$;

6$_\blacktriangle$. If $e \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}((p, \alpha, p', Q)_\mathrm{d}^\eta)$, then there is $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}_\blacktriangle((p, \alpha, p', Q)_\mathrm{d}^\eta)$ with $\mathsf{expr}_\blacktriangle^\varepsilon(\bigwedge\{(\alpha)\varphi'\} \cup \Psi) \leq e$.

7$_\blacktriangle$. If $e \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, Q]_\mathrm{a}^\eta)$ and $e_3 = \infty$, then there is $\varphi \in \mathsf{Strat}_\blacktriangle([p, Q]_\mathrm{a}^\eta)$ with price $\mathsf{expr}_\blacktriangle(\varphi) \leq \mathsf{upd}(e, -\hat{\mathbf{e}}_1)$.

We apply induction over game positions g and energies e according to the inductive Definition 4.3. We distinguish cases depending on the kind of position, and we only repeat the cases that are different from Lemma 5.2.

2$_\blacktriangle$. Assume $e \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, Q]_\mathrm{a}^\varepsilon)$. This must be due to one of the following moves:

Branch. conjunction move $[p, Q]_\mathrm{a}^\varepsilon \overset{\mathbf{0}}{\rightarrowtail} (p, \alpha, p', Q)_\mathrm{d}^\eta$**:** It must hold that $e \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}((p, \alpha, p', Q)_\mathrm{d}^\eta)$. By induction hypothesis there is a formula $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}_\blacktriangle((p, \alpha, p', Q)_\mathrm{d}^\eta)$ and $\mathsf{expr}_\blacktriangle^\varepsilon(\bigwedge\{(\alpha)\varphi'\} \cup \Psi) \leq e$; therefore, by rule (branch) of Definition 7.2, $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}_\blacktriangle([p, Q]_\mathrm{a}^\varepsilon)$.

Other moves: same as for Lemma 5.2.

6$_\blacktriangle$. Assume $e \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}((p, \alpha, p', Q)_\mathrm{d}^\eta)$. Then there are moves $(p, \alpha, p', Q)_\mathrm{d}^\eta \xrightarrow{-\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3} [p, \alpha, p', q]_\mathrm{a}^\eta$ for every $q \in Q$. For every such state, it must be the case that $e' := \mathsf{upd}(e, -\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, \alpha, p', q]_\mathrm{a}^\eta)$. (Note that this implies that $e_3 = \infty$ and therefore $e'_3 = \infty$.) We know that this must be due to one of the following moves:

Branch. observation move $[p, \alpha, p', q]_\mathrm{a}^\eta \xrightarrow{(\mathtt{min}_{\{1,6\}}, 0,0,0,0,0,0,0)} [p', Q']_\mathrm{a}^\eta$ **for** $\{q\} \xrightarrow{(\alpha)} Q'$**:** It must be the case that $e'' := \mathsf{upd}(e', (\mathtt{min}_{\{1,6\}}, 0, 0, 0, 0, 0, 0, 0)) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p', Q']_\mathrm{a}^\eta)$ and $e''_3 = \infty$. By induction hypothesis there is some formula $\varphi \in \mathsf{Strat}_\blacktriangle([p', Q']_\mathrm{a}^\eta, e'')$ and $\mathsf{expr}_\blacktriangle(\varphi) \leq \mathsf{upd}(e'', -\hat{\mathbf{e}}_1)$.

Let Q_α be the subset of Q consisting of states where the attacker chooses the branching observation move, and pick a formula $\psi_{q_\alpha} \in \mathsf{Strat}_\blacktriangle([p', Q']_\mathrm{a}^\eta, e'')$ with $\mathsf{expr}_\blacktriangle(\psi_{q_\alpha}) \leq \mathsf{upd}(e'', -\hat{\mathbf{e}}_1)$ for every $q_\alpha \in Q_\alpha$. Let $\varphi_\alpha := \mathrm{merge}(\{\psi_{q_\alpha} \mid q_\alpha \in Q_\alpha\})$.

Branch. reset move $[p, \alpha, p', q]_\mathrm{a}^\eta \overset{\mathbf{0}}{\rightarrowtail} [p, q]_\mathrm{a}^\wedge$: It must be the case that $e' \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, q]_\mathrm{a}^\wedge)$. By induction hypothesis there is some formula $\psi \in \mathsf{Strat}_\blacktriangle([p, q]_\mathrm{a}^\wedge, e')$ with $\mathsf{expr}_\blacktriangle^\wedge(\psi) \leq e'$.

Now we can apply rule (branch conj) in Definition 7.2 and get that $\bigwedge(\{(\alpha)\varphi_\alpha\} \cup \{\psi_q \mid q \in Q \setminus Q_\alpha\}) \in \mathsf{Strat}_\blacktriangle((p, \alpha, p', Q)_\mathrm{d}^\eta, e)$. When we look through Definition 3.3 of expr, we see that the expressiveness price of this formula is also $\leq e$.

7$_\blacktriangle$. Assume $e \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, Q]_\mathrm{a}^\eta)$. This must be due to one of the following moves:

Early branch. accounting $[p, Q]_\mathrm{a}^\eta \xrightarrow{-\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_5} (p, Q)_\mathrm{d}$**:** We have $e' = \mathsf{upd}(e, -\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_5) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}((p, Q)_\mathrm{d})$, so by induction hypothesis we know that there exists a conjunction $\bigwedge\Psi \in \mathsf{Strat}_\blacktriangle((p, Q)_\mathrm{d}, e')$ and $\mathsf{expr}_\blacktriangle^\varepsilon(\bigwedge\Psi) \leq e'$. But then, $\bigwedge\Psi \in \mathsf{Strat}_\blacktriangle([p, Q]_\mathrm{a}^\eta, e)$ by rule (branch immediate acct) of Definition 7.2 and $\mathsf{expr}_\blacktriangle(\bigwedge\Psi) \leq e' + \hat{\mathbf{e}}_5 = \mathsf{upd}(e, -\hat{\mathbf{e}}_1)$.

Early branch. finishing $[p, \varnothing]_\mathrm{a}^\eta \overset{\mathbf{0}}{\rightarrowtail} (p, \varnothing)_\mathrm{d}$**:** Similar to the previous case.

Late branch. accounting $[p, Q]_\mathrm{a}^\eta \xrightarrow{-\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3} [p, Q']_\mathrm{a}^\varepsilon$ **for** $Q \twoheadrightarrow Q'$**:** It must hold that $e' = \mathsf{upd}(e, -\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3) \in \mathsf{Win}_\mathrm{a}^{\mathcal{G}_\blacktriangle}([p, Q']_\mathrm{a}^\varepsilon)$, so by induction hypothesis we know that there exists a formula $\chi \in \mathsf{Strat}_\blacktriangle([p, Q']_\mathrm{a}^\varepsilon, e')$ and $\mathsf{expr}_\blacktriangle^\varepsilon(\chi) \leq e'$. But then, $\langle\varepsilon\rangle\chi \in \mathsf{Strat}_\blacktriangle([p, Q]_\mathrm{a}^\eta, e)$ by rule (branch late acct) of Definition 7.2 and $\mathsf{expr}_\blacktriangle(\langle\varepsilon\rangle\chi) = e' = \mathsf{upd}(e, -\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3) = \mathsf{upd}(e, -\hat{\mathbf{e}}_1)$, using that $e_3 = \infty$ in the last equality. □

Lemma 7.3. *If $\varphi \in \mathsf{Strat}_\blacktriangle([p, Q]_\mathrm{a}, e)$, then φ distinguishes p from Q.*

Proof. Again, to get an inductive property, we actually prove the following:

1. If $\varphi \in \mathsf{Strat}_\blacktriangle([p, Q]_\mathrm{a}, e)$, then φ distinguishes p from Q;

2$_\blacktriangle$. If $\chi \in \mathsf{Strat}_\blacktriangle([p, Q]_\mathrm{a}^\varepsilon, e)$ and $Q \twoheadrightarrow Q$, then $\langle\varepsilon\rangle\chi$ distinguishes p from Q;

3. If $\psi \in \mathsf{Strat}_\blacktriangle([p,q]^{\wedge}_{\mathrm{a}}, e)$, then ψ distinguishes p from $\{q\}$.

4. If $\bigwedge\Psi \in \mathsf{Strat}_\blacktriangle((p,Q)_{\mathrm{d}}, e)$, then $\bigwedge\Psi$ distinguishes p from Q;

5. If $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi \in \mathsf{Strat}_\blacktriangle((p,Q)^{s}_{\mathrm{d}}, e)$ and p is stable, then the stable conjunction $\bigwedge\{\neg\langle\tau\rangle\mathsf{T}\} \cup \Psi$ distinguishes p from Q;

6$_\blacktriangle$. If $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}_\blacktriangle((p,\alpha,p',Q)^{\eta}_{\mathrm{d}}, e)$ and $p \xrightarrow{(\alpha)} p'$, then the branching conjunction $\bigwedge\{(\alpha)\varphi'\} \cup \Psi$ distinguishes p from Q.

7$_\blacktriangle$. If $\varphi \in \mathsf{Strat}_\blacktriangle([p,Q]^{\eta}_{\mathrm{a}}, e)$, then φ distinguishes p from Q.

We prove the result by induction over the derivation of $\cdots \in \mathsf{Strat}_\blacktriangle(g, e)$ according to Definition 7.2.

2$_\blacktriangle$. Assume $\chi \in \mathsf{Strat}_\blacktriangle([p,Q]^{\varepsilon}_{\mathrm{a}}, e)$ and $Q \twoheadrightarrow Q$.

Due to rule (branch) in Definition 7.2: Then $\chi \in \mathsf{Strat}_\blacktriangle((p,\alpha,p',Q)^{\eta}_{\mathrm{d}}, e)$ (for some $p \xrightarrow{(\alpha)} p'$). By induction hypothesis, χ distinguishes p from Q. So $p \in [\![\chi]\!] \subseteq [\![\langle\varepsilon\rangle\chi]\!]$. If there were some $q \in Q \cap [\![\langle\varepsilon\rangle\chi]\!]$, then we would have a path $q \twoheadrightarrow q' \in [\![\chi]\!]$. But $q' \in Q$ because $Q \twoheadrightarrow Q$, and this contradicts that χ distinguishes p from Q. Therefore $\langle\varepsilon\rangle\chi$ distinguishes p from Q.

Due to other rules: Same as in Lemma 5.3.

6$_\blacktriangle$. Assume $\bigwedge\{(\alpha)\varphi'\} \cup \Psi \in \mathsf{Strat}_\blacktriangle((p,\alpha,p',Q)^{\eta}_{\mathrm{d}}, e)$ and $p \xrightarrow{(\alpha)} p'$.

Due to rule (branch conj) in Definition 7.2: Then Ψ can be written as $\{\psi_q \mid q \in Q \setminus Q_\alpha\}$, where $\psi_q \in \mathsf{Strat}_\blacktriangle([p,q]^{\wedge}_{\mathrm{a}}, \mathsf{upd}(e, -\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3))$. For every $q_\alpha \in Q_\alpha$, by the definition of Q_α in rule (branch conj) of Definition 7.2, there exists a set Q'_{q_α} such that $\{q_\alpha\} \xrightarrow{(\alpha)} Q'_{q_\alpha}$ and $\psi_{q_\alpha} \in \mathsf{Strat}_\blacktriangle([p',Q'_{q_\alpha}]^{\eta}_{\mathrm{a}}, \ldots)$. By induction hypothesis, ψ_q distinguishes p from q. Because this holds for every $q \in Q \setminus Q_\alpha$, we have that $\bigwedge\Psi$ distinguishes p from $Q \setminus Q_\alpha$.

Moreover, there are moves $(p,\alpha,p',Q)^{\eta}_{\mathrm{d}} \rightarrowtail [p,\alpha,p',q_\alpha]^{\eta}_{\mathrm{a}} \rightarrowtail [p',Q'_{q_\alpha}]^{\eta}_{\mathrm{a}}$ for every $q_\alpha \in Q_\alpha$, for some set Q'_{q_α}. We have $p \xrightarrow{(\alpha)} p'$ and $\{q_\alpha\} \xrightarrow{(\alpha)} Q'_{q_\alpha}$, and also $\psi_{q_\alpha} \in \mathsf{Strat}_\blacktriangle([p',Q'_{q_\alpha}]^{\eta}_{\mathrm{a}}, \mathsf{upd}(\mathsf{upd}(e, -\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3), (\mathtt{min}_{\{1,6\}}, 0,0,0,0,0,0,0)))$. By induction hypothesis, ψ_{q_α} distinguishes p' from Q'_{q_α}. The formula $\varphi_\alpha := \mathrm{merge}(\{\psi_{q_\alpha} \mid q_\alpha \in Q_\alpha\})$ therefore distinguishes p' from every such Q'_{q_α}, and then $(\alpha)\varphi_\alpha$ distinguishes p from Q_α.

Together we have that $\bigwedge\{(\alpha)\varphi_\alpha\} \cup \Psi$ distinguishes p from Q.

7$_\blacktriangle$. Assume $\varphi \in \mathsf{Strat}_\blacktriangle([p,Q]^{\eta}_{\mathrm{a}}, e)$.

Due to rule (branch immediate acct) in Definition 7.2: Then $\varphi \in \mathsf{Strat}_\blacktriangle((p,Q)_{\mathrm{d}}, \mathsf{upd}(e,u))$ for $u = -\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_5$ (or $u = -\hat{\mathbf{e}}_1$ if $Q = \varnothing$). By induction hypothesis, φ distinguishes p from Q, which is exactly what needs to be proven.

Due to rule (branch late acct) in Definition 7.2: Then $\varphi = \langle\varepsilon\rangle\chi$ and $\chi \in \mathsf{Strat}_\blacktriangle([p,Q']^{\varepsilon}_{\mathrm{a}}, \mathsf{upd}(e, -\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3))$. Also, $Q \twoheadrightarrow Q'$ and therefore $Q' \twoheadrightarrow Q'$. By induction hypothesis (case 2$_\blacktriangle$), $\langle\varepsilon\rangle\chi$ distinguishes p from Q', so $\langle\varepsilon\rangle\chi$ also distinguishes p from $Q \subseteq Q'$. □

8. Related Work and Conclusion

This paper provides the first *generalized game characterization* for the spectrum of *"weak" behavioral equivalences* and preorders. To this end, Section 2 has introduced a new *modal characterization of branching bisimilarity* that is used to capture the *modal logics of the silent-step spectrum* in Section 3. With this perspective, the family of weak equivalence problems becomes just *one quantitative problem,* expressible as one energy game in Section 4. As we saw in Section 6, this can be used to conveniently solve research tasks in concurrency theory, and could be extended to further notions. Thanks to the game simplification of Section 7, meaningful practical models with considerable internal non-determinism can be handled.

Other *generalized game characterizations* by Chen and Deng [27] and by us [8, 9] have only addressed strong equivalences or parts of the spectrum [28, 29]. Fahrenberg et al. [30] treated a quantitative game interpretation for behavioral distances, as well disregarding silent-step notions. Extending this line of work to account for silent steps in full is necessary for virtually every application.

In the silent-step spectrum, many things are more complicated. There are *several abstractions of bisimilarity:* branching, η, delay and weak bisimilarity, as well as contrasimilarity, stable bisimilarity and coupled similarity. We have had to radically depart from their existing games [14, 15, 31] to cover all equivalences. Depending on *whether stabilization is required* for negated and conjunct observations, each equivalence notion has different weak versions. Our game characterization is the first to explicitly consider stability-respecting notions, thereby unifying stable equivalences [6] and unstable ones [13]. This unification enables observations about the applicability of (un)stable equivalences as the one in Example 6.1.

The *framework of codesigning games and grammars* can also easily be extended to cater for more notions, for instance, divergence-aware ones as hinted to in Section 6, or even to combine strong and weak ones in one game. The connection to energy games enables us to boost our approach using recent polynomial decision procedures for multi-weighted games from Lemke [23] and Brihaye and Goeminne's [32].

We have added to the rich body of work on *modal characterizations of branching bisimilarity* [33, 6, 19, 34, 18]. Continuing [8, 9], our work participates in a recent trend towards a modal focus for equivalences, also found in Ford et al. [35] connecting graded modal logics and monads, and in Wißmann et al. [36] as well as Beohar et al. [37]. Like Martens and Groote [38], we find minimal-depth distinguishing formulas for branching bisimilarity, but we solve the problem for all weak notions at once.

Our main related work, of course, is van Glabbeek's *linear-time–branching-time spectrum* [5, 6]. Up to today, part II on silent steps is available only in preliminary versions. For our slice of the weak spectrum, we have filled in some blanks on modal characterizations. Most importantly, we hope our algorithms make the wisdom on weak equivalences of part II more accessible to tools and humans alike.

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References

[1] B. Bisping, D. N. Jansen, One energy game for the spectrum between branching bisimilarity and weak trace semantics, in: G. Caltais, C. Di Giusto (Eds.), Proceedings Combined 31st International Workshop on Expressiveness in Concurrency and 21st Workshop on Structural Operational Semantics, Calgary, Canada, 9th September 2024, Vol. 412 of Electronic Proceedings in Theoretical Computer Science, Open Publishing Association, 2024, pp. 71–88. doi:10.4204/EPTCS.412.6.

[2] C. J. Bell, Certifiably sound parallelizing transformations, in: G. Gonthier, M. Norrish (Eds.), Certified Programs and Proofs: CPP, Vol. 8307 of LNCS, Springer, Cham, 2013, pp. 227–242. doi:10.1007/978-3-319-03545-1_15.

[3] A. D. Barwell, F. Ferreira, N. Yoshida, CONCUR test-of-time award for the period 1994–97 interview with Uwe Nestmann and Benjamin C. Pierce, Journal of Logical and Algebraic Methods in Programming 125 (2022) 100744. doi:10.1016/j.jlamp.2021.100744.

[4] R. Horne, S. Mauw, Discovering ePassport vulnerabilities using bisimilarity, Logical Methods in Computer Science 17 (2) (2021) 24:1–24:52. doi:10.23638/LMCS-17(2:24)2021.

[5] R. van Glabbeek, The linear time–branching time spectrum: extended abstract, in: J. C. M. Baeten, J. W. Klop (Eds.), CONCUR'90, Vol. 458 of LNCS, Springer, Berlin, 1990, pp. 278–297. doi:10.1007/BFb0039066.

[6] R. van Glabbeek, The linear time–branching time spectrum II: The semantics of sequential systems with silent moves; extended abstract, in: E. Best (Ed.), CONCUR'93, Vol. 715 of LNCS, Springer, Berlin, 1993, pp. 66–81. doi:10.1007/3-540-57208-2_6.

[7] R. van Glabbeek, The linear time–branching time spectrum I: The semantics of concrete, sequential processes, in: J. A. Bergstra, A. Ponse, S. A. Smolka (Eds.), Handbook of Process Algebra, Elsevier, Amsterdam, 2001, Ch. 1, pp. 3–99. doi:10.1016/B978-044482830-9/50019-9.

[8] B. Bisping, D. N. Jansen, U. Nestmann, Deciding all behavioral equivalences at once: A game for linear-time–branching-time spectroscopy, Logical Methods in Computer Science 18 (3) (2022) 19:1–19:33. doi:10.46298/lmcs-18(3:19)2022.

[9] B. Bisping, Process equivalence problems as energy games, in: C. Enea, A. Lal (Eds.), Computer Aided Verification, Springer Nature Switzerland, Cham, 2023, pp. 85–106. doi:10.1007/978-3-031-37706-8_5.

[10] C. Lemke, B. Bisping, Galois Energy Games: To Solve All Kinds of Quantitative Reachability Problems, in: P. Bouyer, J. van de Pol (Eds.), 36th International Conference on Concurrency Theory (CONCUR 2025), Vol. 348 of Leibniz International Proceedings in Informatics (LIPIcs), Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2025, pp. 29:1–29:19. doi:10.4230/LIPIcs.CONCUR.2025.29.

[11] U. Fahrenberg, L. Juhl, K. G. Larsen, J. Srba, Energy games in multiweighted automata, in: A. Cerone, P. Pihlajasaari (Eds.), Theoretical Aspects of Computing – ICTAC 2011, Vol. 6916 of LNCS, Springer, Heidelberg, 2011, pp. 95–115. doi:10.1007/978-3-642-23283-1_9.

[12] O. Kupferman, N. Shamash Halevy, Energy games with resource-bounded environments, in: B. Klin, S. Lasota, A. Muscholl (Eds.), 33rd International Conference on Concurrency Theory: CONCUR, Vol. 243 of LIPIcs, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Saarbrücken, 2022, pp. 19:1–19:23. doi:10.4230/LIPIcs.CONCUR.2022.19.

[13] M. Gazda, W. Fokkink, V. Massaro, Congruence from the operator's point of view: Syntactic requirements on modal characterizations, Acta Informatica 57 (3–5) (2020) 329–351. doi:10.1007/s00236-019-00355-5.

[14] D. de Frutos Escrig, J. J. A. Keiren, T. A. C. Willemse, Games for bisimulations and abstraction, Logical Methods in Computer Science 13 (4:15) (2017) 1–40. doi:10.23638/LMCS-13(4:15)2017.

[15] B. Bisping, U. Nestmann, K. Peters, Coupled similarity: the first 32 years, Acta Informatica 57 (3–5) (2020) 439–463. doi:10.1007/s00236-019-00356-4.

[16] L. A. Barthel, L. M. Hübner, C. Lemke, K. P. P. Mattes, L. Mollenkopf, B. Bisping, The weak spectroscopy game to characterize behavioral equivalences, Archive of Formal Proofs (11 2025).
URL https://isa-afp.org/entries/Weak_Spectroscopy.html

[17] B. Bisping, Generalized equivalence checking of concurrent programs, Ph.D. thesis, Technische Universität Berlin (2025). doi:10.14279/depositonce-24813.
URL https://generalized-equivalence-checking.equiv.io/

[18] H. Geuvers, Apartness and distinguishing formulas in Hennessy–Milner logic, in: N. Jansen, M. Stoelinga, P. van den Bos (Eds.), A journey from process algebra via timed automata to model learning: essays dedicated to Frits Vaandrager on the occasion of his 60th birthday, Vol. 13560 of LNCS, Springer, Cham, 2022, pp. 266–282. doi:10.1007/978-3-031-15629-8_14.

[19] W. Fokkink, R. van Glabbeek, B. Luttik, Divide and congruence III: From decomposition of modal formulas to preservation of stability and divergence, Information and Computation 268 (2019) 104435. doi:10.1016/j.ic.2019.104435.

[20] D. de Frutos Escrig, C. Gregorio Rodríguez, M. Palomino, D. Romero Hernández, Unifying the Linear Time-Branching Time Spectrum of Strong Process Semantics, Logical Methods in Computer Science 9 (2:11) (2013) 1–74. doi:10.2168/LMCS-9(2:11)2013.

[21] B. Bisping, L. Montanari, Characterizing contrasimilarity through games, modal logic, and complexity, Inf. Comput. 300 (2024) 105191. doi:10.1016/J.IC.2024.105191.

[22] C. Lemke, Galois energy games, Archive of Formal ProofsFormal Isabelle/HOL proof development (2025).
URL https://isa-afp.org/entries/Galois_Energy_Games.html

[23] C. Lemke, A formal proof of decidability of multi-weighted declining energy games, Master's thesis, Technische Universität Berlin (2024).
URL https://github.com/crmrtz/galois-energy-games/blob/e0355ce5bd66f1f388b1c496cea4a1441d6d8677/master-thesis/masters_thesis.pdf

[24] J. R. Andersen, N. Andersen, S. Enevoldsen, M. M. Hansen, K. G. Larsen, S. R. Olesen, J. Srba, J. K. Wortmann, CAAL: Concurrency workbench, Aalborg edition, in: M. Leucker, C. Rueda, F. D. Valencia (Eds.), Theoretical Aspects of Computing – ICTAC 2015, Springer International Publishing, Cham, 2015, pp. 573–582. doi:10.1007/978-3-319-25150-9_33,.

[25] A. W. Roscoe, Revivals, stuckness and the hierarchy of CSP models, Journal of Logic and Algebraic Programming 78 (3) (2009) 163–190. doi:10.1016/j.jlap.2008.10.002.

[26] J. F. Groote, D. N. Jansen, J. J. A. Keiren, A. J. Wijs, An $\mathcal{O}(m \log n)$ algorithm for computing stuttering equivalence and branching bisimulation, ACM Transactions on Computational Logic (TOCL) 18 (2) (2017) 13:1–13:34. doi:10.1145/3060140.

[27] X. Chen, Y. Deng, Game characterizations of process equivalences, in: G. Ramalingam (Ed.), Programming Languages and Systems: APLAS, Vol. 5356 of LNCS, Springer, Berlin, 2008, pp. 107–121. doi:10.1007/978-3-540-89330-1_8.

[28] S. K. Shukla, H. B. Hunt III, D. J. Rosenkrantz, HORNSAT, model checking, verification and games: extended abstract, in: R. Alur, T. A. Henzinger (Eds.), Computer Aided Verification: CAV, Vol. 1102 of LNCS, Springer, Berlin, 1996, pp. 99–110. doi:10.1007/3-540-61474-5_61.

[29] L. Tan, An abstract schema for equivalence-checking games, in: A. Cortesi (Ed.), Verification, Model Checking, and Abstract Interpretation, Third International Workshop, VMCAI 2002, Venice, Italy, January 21-22, 2002, Revised Papers, Vol. 2294 of Lecture Notes in Computer Science, Springer, 2002, pp. 65–78. doi:10.1007/3-540-47813-2_5.

[30] U. Fahrenberg, A. Legay, The quantitative linear-time–branching-time spectrum, Theoretical Computer Science 538 (2014) 54–69, quantitative Aspects of Programming Languages and Systems (2011-12). doi:10.1016/j.tcs.2013.07.030.

[31] B. Bisping, L. Montanari, A game characterization for contrasimilarity, in: O. Dardha, V. Castiglioni (Eds.), Proceedings Combined 28th International Workshop on Expressiveness in Concurrency and 18th Workshop on Structural Operational Semantics, Vol. 339 of Electronic Proceedings in Theoretical Computer Science, Open Publishing Association, Waterloo, Australia, 2021, pp. 27–42. doi:10.4204/EPTCS.339.5.

[32] T. Brihaye, A. Goeminne, Multi-weighted reachability games, in: O. Bournez, E. Formenti, I. Potapov (Eds.), Reachability Problems, RP 2023, Springer Nature Switzerland, Cham, 2023, pp. 85–97. doi:10.1007/978-3-031-45286-4_7.

[33] R. De Nicola, F. Vaandrager, Three logics for branching bisimulation, J. ACM 42 (2) (1995) 458–487. doi:10.1145/201019.201032.

URL https://doi.org/10.1145/201019.201032

[34] H. Geuvers, A. Golov, Positive Hennessy-Milner logic for branching bisimulation (2023). arXiv:2210.07380.

[35] C. Ford, S. Milius, L. Schröder, Behavioural preorders via graded monads, in: 2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), IEEE, New York, NY, USA, 2021, pp. 1–13. doi:10.1109/LICS52264.2021.9470517.

[36] T. Wißmann, S. Milius, L. Schröder, Explaining Behavioural Inequivalence Generically in Quasilinear Time, in: S. Haddad, D. Varacca (Eds.), 32nd International Conference on Concurrency Theory (CONCUR 2021), Vol. 203 of Leibniz International Proceedings in Informatics (LIPIcs), Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2021, pp. 32:1–32:18. doi:10.4230/LIPIcs.CONCUR.2021.32.

[37] H. Beohar, S. Gurke, B. König, K. Messing, Hennessy-Milner Theorems via Galois Connections, in: B. Klin, E. Pimentel (Eds.), 31st EACSL Annual Conference on Computer Science Logic (CSL 2023), Vol. 252 of Leibniz International Proceedings in Informatics (LIPIcs), Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2023, pp. 12:1–12:18. doi:10.4230/LIPIcs.CSL.2023.12.

[38] J. Martens, J. F. Groote, Minimal depth distinguishing formulas without until for branching bisimulation, in: V. Capretta, R. Krebbers, F. Wiedijk (Eds.), Logics and Type Systems in Theory and Practice: Essays Dedicated to Herman Geuvers on The Occasion of His 60th Birthday, Springer Nature, Cham, 2024, pp. 188–202. doi:10.1007/978-3-031-61716-4_12.